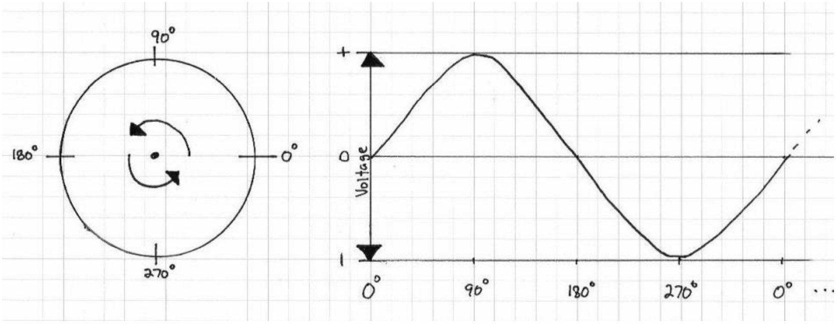
ELEC 2400 Electronic Circuits

Chapter 3: AC Steady-State Analysis



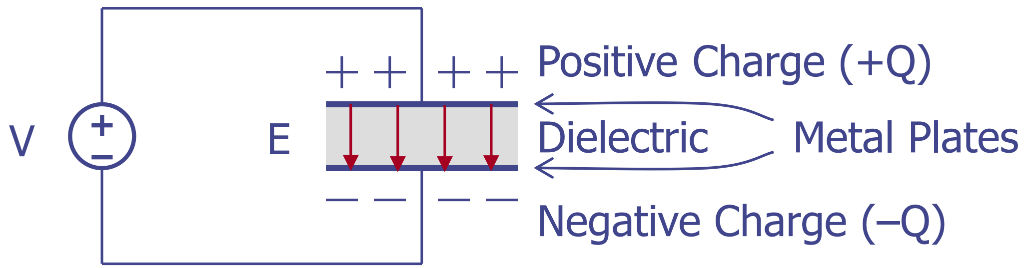
Course Website: https://canvas.ust.hk

HKUST, 2021-22 Fall

Chapter 3: AC Steady-State Analysis

3.1.1 Capacitor

Charge can be stored on the surface of a conductor that is surrounded by insulator. The circuit element that is used to store charge is the capacitor. A capacitor can be formed by using two metal plates separated by a dielectric material (insulator) (parallel plate capacitor).



The amount of charge stored is proportional to voltage, and is given by

$$Q = CV$$

with "Q" understood as having +Q on the positive plate and –Q on the negative plate, and C is the capacitance, and the unit is farad (F), with the dimension of [coulomb/volt].

Capacitance

The capacitance of the parallel plate capacitor can be derived from:

- $Q = \varepsilon EA$ (Gauss's Law)
- $E = V/d$
- $Q = CV$

Michael Faraday
1791 - 1867

Result is:

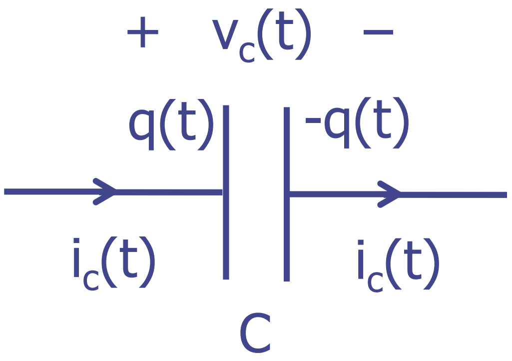
$$C = \frac{\varepsilon A}{d} = \frac{k\varepsilon_o A}{d}$$

where

- C = capacitance (in farad, F)
- Q = charge (in coulomb, C)
- V = voltage (in volt, V)
- E = electric field (in V/m)
- ε_o = permittivity of free space (vacuum) $= 8.854\times10^{-12}$ F/m
- ε = $k\varepsilon_o$ = permittivity of dielectric material
- k = dielectric constant (relative permittivity)
- d = distance between plates
- A = cross-sectional area of plates

Example 3-1: Mica capacitor has $k = 5$. If $A = 0.5$ cm$\times$0.5 cm, $d = 100$ μm, then $C = 5\times8.85\times10^{-12}\times0.005\times0.005/100\times10^{-6} = 11$ pF.

Capacitor Voltage and Current Relationship



Fundamental equation

$$q(t) = Cv_c(t)$$

Current is the change in charge over time, and the differential form gives the (time domain) I-V relationship of the capacitor:

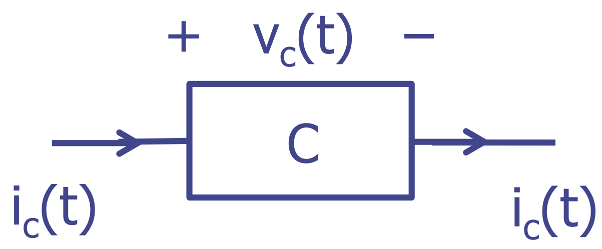
$$i_c(t) = \frac{dq(t)}{dt} = C\frac{dv_c(t)}{dt}$$

Integrating once, we get:

$$v_c(t) = v_c(0) + \frac{1}{C}\int_0^t i_c(\tau)d\tau$$

Capacitor Blocks DC, Passes AC?

The capacitor dielectric, which is an insulator, allows no moving charges to pass through it. A capacitor therefore passes no current whatsoever: DC, AC or transient.



However, modeled as a black box and viewed from external, the capacitor does give an illusion that a current passes through it, which could be a useful way of reasoning.

A capacitor eventually blocks a DC current because the capacitor voltage cannot increase forever. For a DC circuit, the capacitor is charged to a max. voltage set by the circuit. After which current stops and the capacitor behaves like an open circuit.

Energy Stored in Capacitor

The energy stored in a capacitor is dependent on its charge Q, voltage V, and capacitance C.

To move an infinitesimal charge dq from the negative plate to the positive plate (from a lower to a higher potential), the amount of work dW that must be done on dq is $dW = v dq$.

This work becomes the energy stored in the electric field of the capacitor. In order to charge the capacitor to a charge from 0 to Q, the total work required is

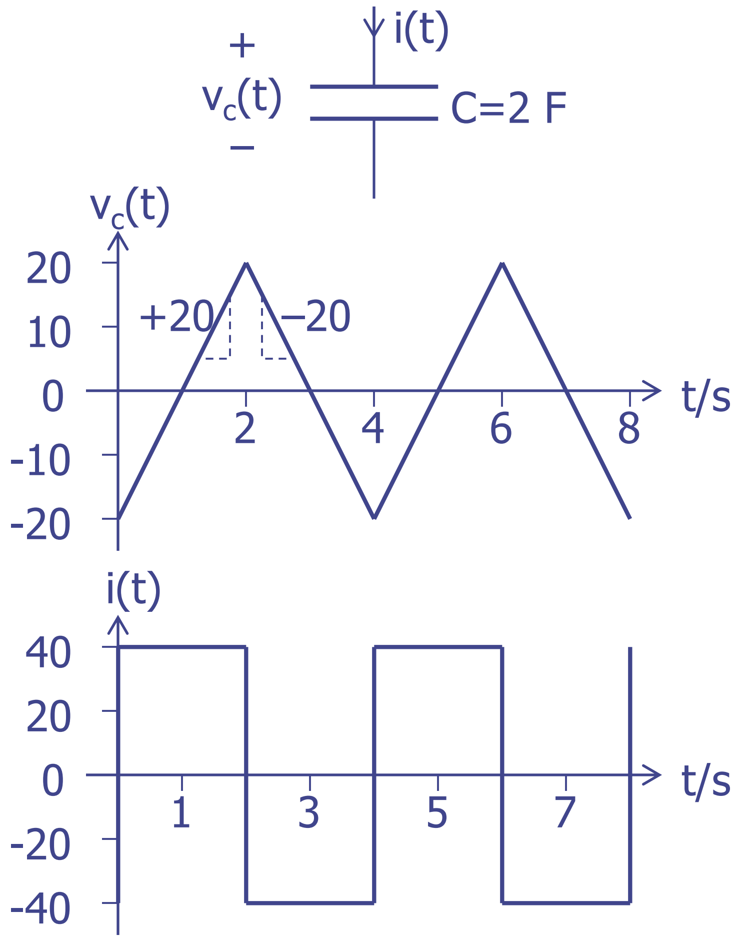
$$W = \int_0^{W(Q)} dW = \int_0^Q v dq = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C}$$

Hence the energy stored in a capacitor is

$$E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

Example 3-2

Example 3-2: Given $v_c(t)$ across C as shown, find i(t) and i(1 s).



Soln. For t < 2 s:

$$i(t) = C\frac{dv_c(t)}{dt}$$

$$= 2 \times \frac{20}{1}$$

$$= 40A$$

Similarly, for 2 s < t < 4 s:

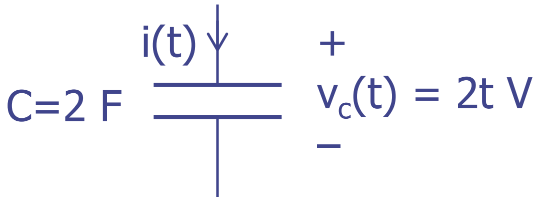
$$i(t) = -40A$$

and

$$i(1s) = 40A$$

Examples 3-3, 3-4

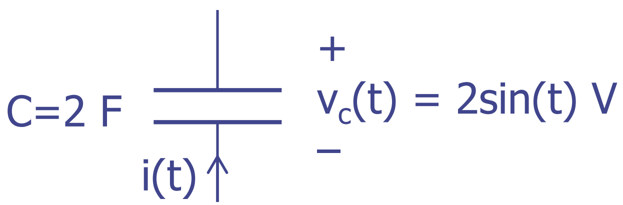
Example: 3-3: Given $v_c(t)$ across C as shown, find i(t).



Soln:

i(t) = 2 F×2 V/s = 4 A

Example: 3-4: Given $v_c(t)$ across C as shown, find i(t).



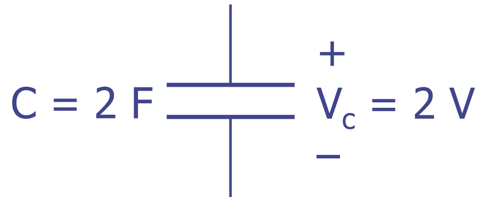
Soln:

$$i(t) = -2 \times \frac{d(2\sin(t))}{dt} = -2 \times 2\cos(t)$$

$$= -4\cos(t) \text{ A}$$

Examples 3-5, 3-6

Example: 3-5: Find the energy stored in the capacitor below.



Soln:

$$E_c = \frac{1}{2}CV_c^2 = \frac{1}{2} \times 2 \times 2^2$$
$$= 4\ \text{J}$$

Example: 3-6: For C = 2 F, find $v_c(1\text{ s})$ given that $V_c(0) = -20$ V and $i_c(t) = 40$ A.

Soln:

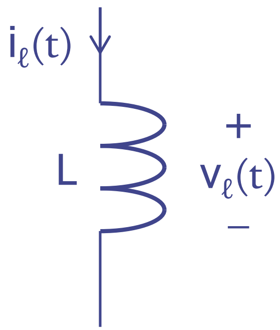
$$v_c(t) = v_c(0) + \frac{1}{C}\int_0^t i_c(\tau)d\tau$$
$$= -20 + \frac{1}{2}\int_0^1 40 d\tau$$
$$= -20 + 20 = 0\text{V}$$

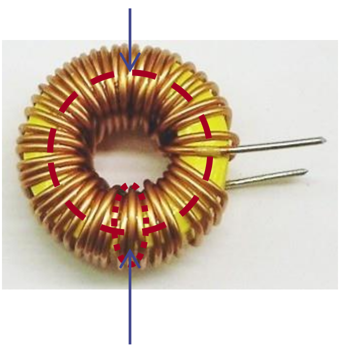
3.1.2 Inductor

When current passes through a medium, magnetic flux ϕ, and the unit is weber (Wb), is produced that bears the relation

$$\phi(t) = L \times i(t)$$

where L is the inductance, and the unit is henry (H). The circuit element that stores magnetic flux (magnetic energy) is the inductor. An inductor can be formed by wrapping a coil around a ferromagnetic material.





Faraday's Law of Induction:

$$v_\ell(t) = \frac{d\phi(t)}{dt}$$

and

$$v_\ell(t) = \frac{d\phi(t)}{di_\ell} \cdot \frac{di_\ell}{dt}$$

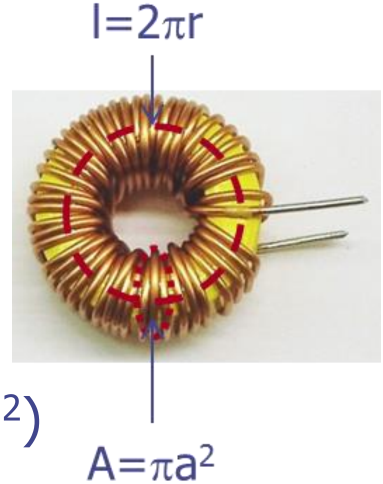
$$= L\frac{di_\ell}{dt}$$

Integrating once, $$i_\ell(t) = i_\ell(0) + \frac{1}{L}\int_0^t v_\ell(\tau)d\tau$$

Inductance

Let the coil has N turns. When the current $i_\ell(t)$ passes through the coil, the effective current is increased by N times, and the magnetic flux thus produced is also increased by N times. The inductance L of a coil inductor can be shown to be

$$L = \frac{\mu N^2 A}{l}$$



where N = number of turns
A = cross-section area of magnetic flux ($= \pi a^2$)
l = length of magnetic path ($= 2\pi r$)
μ = permeability of material (H/m)
μ_o = permeability of free space = $4\pi\times10^{-7}$ H/m

Energy Stored in Inductor

Suppose that an inductor of inductance L is connected to a variable DC voltage supply. The supply is adjusted so as to increase the current flowing through the inductor from zero to some final value I. As the current through the inductor is ramped up, a voltage $v_l = L\frac{di_l}{dt}$ appears across the inductor, which acts to oppose the increase in the current. Clearly, work must be done against this voltage by the voltage supply in order to establish the current in the inductor. This work becomes the energy stored in the magnetic field of the inductor. The work done by the voltage supply during a time interval dt is

$$dW = Pdt = v_l i_l dt = \left(L\frac{di_l}{dt}\right) i_l dt = L i_l di_l$$

The total work required is

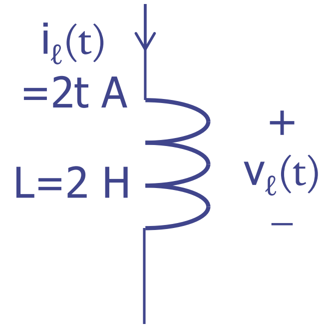
$$W = \int_0^{W(I)} dW = \int_0^I L i_l di_l = \frac{1}{2}LI^2$$

Hence the energy stored in an inductor is

$$E = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I = \frac{1}{2}\frac{\Phi^2}{L}$$

Examples 3-7

Example: 3-7: Find $\phi(t)$, $v_\ell(t)$ and $E_\ell(t)$ of the inductor below.



Soln:

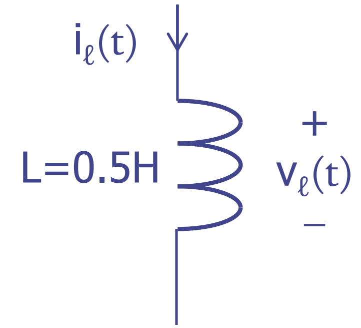
$$\phi(t) = L\times i_\ell(t) = 2\times 2t = 4t \text{ Wb}$$

$$v_\ell(t) = L\frac{di_\ell(t)}{dt} = 2\times 2 = 4V$$

$$E_\ell(t) = \frac{1}{2}Li_\ell(t)^2 = \frac{1}{2}\times 2\times (2t)^2 = 4t^2 \text{ J}$$

Examples 3-8

Example 3-8: Given $i_\ell(t)$ through L as shown, find $v_\ell(t)$ and $v_\ell(1.5s)$.

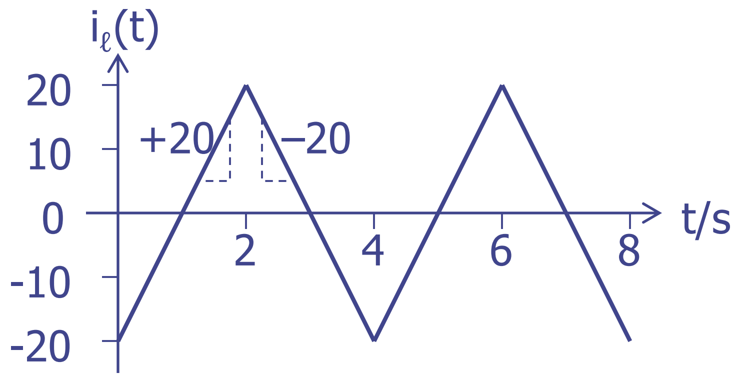


Soln. For t < 2 s:

$$v_\ell(t) = L\frac{di_\ell(t)}{dt}$$

$$= 0.5 \times \frac{20}{1}$$

$$= 10V$$



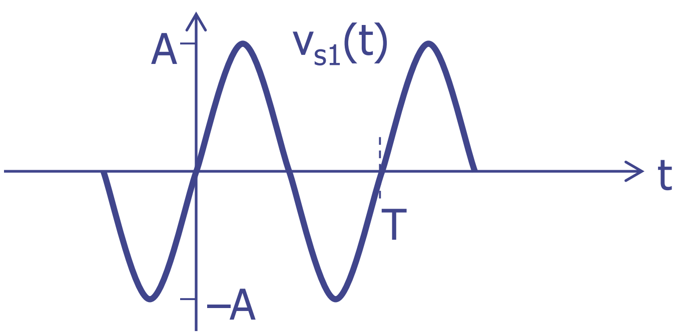
Chapter 3: AC Steady-State Analysis

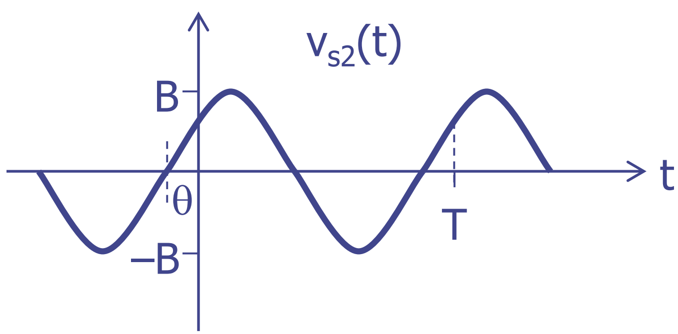
3.2 Sinusoidal Excitation

In DC analysis, the driving forces are DC voltages/currents.

In many other cases, the driving term is a sinusoid, such as the AC (alternating current) voltage obtainable from the wall socket, and we are interested in the sinusoidal (AC) steady-state response. The steady state is the state of the circuit after a long time has elapsed since the application of the sinusoidal source.

A sinusoid wave is characterized by its oscillation frequency, magnitude and phase.





$v_{s1}(t) = A\sin(\omega_1 t)$ $\qquad$ $v_{s2}(t) = B\sin(\omega_2 t+\theta)$

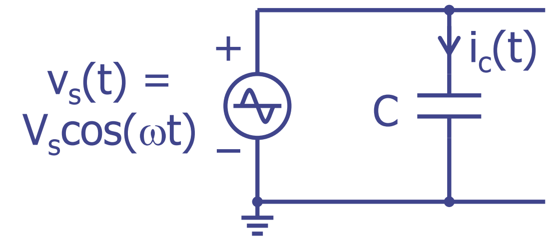
$\omega=2\pi f$ (angular frequency in radian/sec), f (frequency in hertz), T = 1/f (period in sec), θ (phase in radian or degree).

3.2.1 Driving Capacitor with $V_s\cos(\omega t)$

Consider driving a capacitor C with a sinusoidal voltage source:

$$v_s(t) = V_s\cos(\omega t)$$

$$\Rightarrow \quad i_c(t) = C\frac{dv_s(t)}{dt} = -\omega C V_s\sin(\omega t)$$

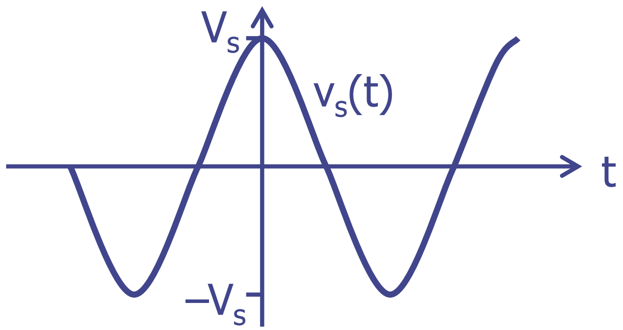


Recall trigonometric identities:

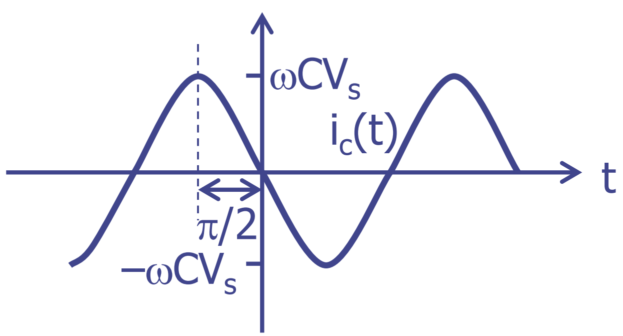
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

Hence,

$$i_c(t) = \omega C V_s\cos(\omega t + \pi/2)$$



The argument of $i_c(t)$ is $+\pi/2$ radian earlier than $v_c(t)$, and the capacitor current leads the capacitor voltage by $\pi/2$ radian (90^o).



3.2.2 Driving Inductor with $V_s\cos(\omega t)$

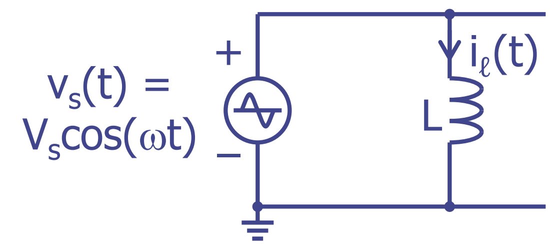
Consider driving an inductor L with a sinusoidal voltage source:

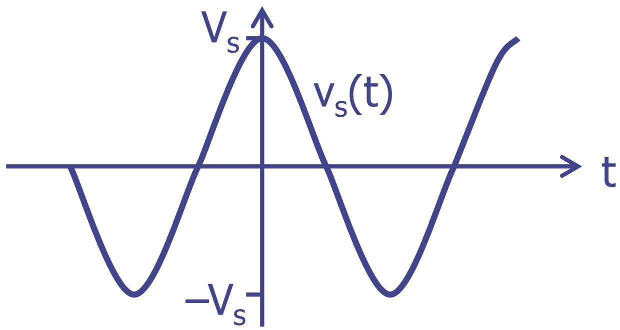
$$v_s(t) = V_s\cos(\omega t)$$

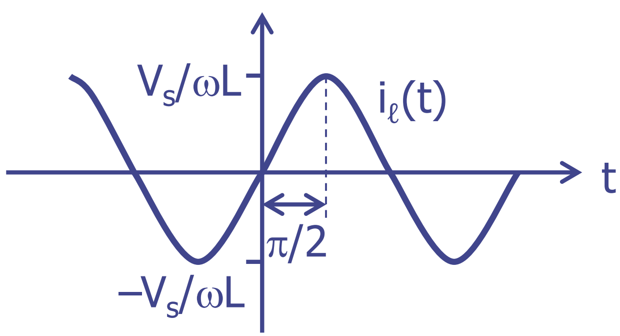
$$\Rightarrow \quad i_\ell(t) = \frac{1}{L}\int v_s(t')dt'$$

$$= \frac{1}{\omega L} \times V_s\sin(\omega t) + \text{constant}^*$$

$$\Rightarrow \quad i_\ell(t) = \frac{1}{\omega L} \times V_s\cos(\omega t - \pi/2)$$







The argument of $i_\ell(t)$ is $\pi/2$ radian later than $v_\ell(t)$, and the inductor current lags the inductor voltage by $\pi/2$ radian (90^o).

* DC component ignored for AC analysis

Example 3-9

Example 3-9: Given an AC voltage source $v_s(t) = V_s\cos(\omega t)$ with amplitude 10 V. Compute the current passing through a capacitor of 10 μF if the frequency is (a) 1 kHz; (b) 10 kHz; and (c) 100 kHz.

Soln.:

$$i_c(t) = C\frac{dv_s(t)}{dt} = 10\mu \times \frac{d(10\cos(2\pi ft))}{dt}$$

$$= -10\mu \times 10 \times 2\pi f \times \sin(2\pi ft)$$

(a) $i_c(t) = -0.628\sin(2\pi 1kt)$ A

(b) $i_c(t) = -6.28\sin(2\pi 10kt)$ A

(c) $i_c(t) = -62.8\sin(2\pi 100kt)$ A

Note that a capacitor serves as an open circuit at low frequency and short circuit at high frequency.

Example 3-10

Example 3-10: Given an AC voltage source $v_s(t) = V_s\cos(\omega t)$ with amplitude 10 V. Compute the current passing through a inductor of 10 μH if the frequency is (a) 1 kHz; (b) 10 kHz; and (c) 100 kHz.

Soln.:

$$i_\ell(t) = \frac{1}{L}\int v_s(t)dt = \frac{1}{10\mu}\int 10\cos(2\pi f t)\,dt$$

$$= \frac{10}{10\mu \times 2\pi f}\sin(2\pi f t)$$

(a) $i_\ell(t) = +159\sin(2\pi 1kt)$ A

(b) $i_\ell(t) = +15.9\sin(2\pi 10kt)$ A

(c) $i_\ell(t) = +1.59\sin(2\pi 100kt)$ A

Note that an inductor serves as a short circuit at low frequency and open circuit at high frequency.

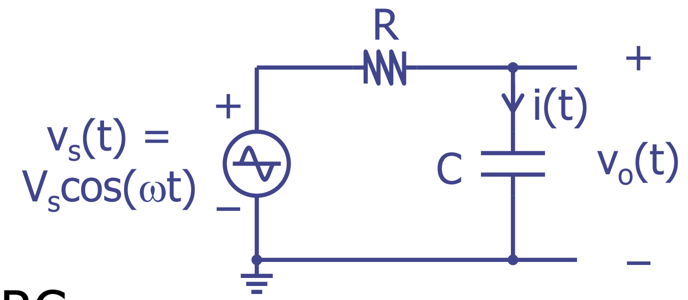
3.2.3 Driving RC Circuit with $V_s\cos(\omega t)$

Consider driving an RC circuit with $v_s(t) = V_s\cos(\omega t)$. KVL gives

$$v_s(t) = Ri(t) + v_o(t)$$

$$\Rightarrow \quad V_s\cos(\omega t) = RC\frac{dv_o(t)}{dt} + v_o(t)$$

$$\Rightarrow \quad \frac{dv_o(t)}{dt} + \frac{1}{\tau}v_o(t) = \frac{V_s}{\tau}\cos(\omega t), \quad \tau = RC$$



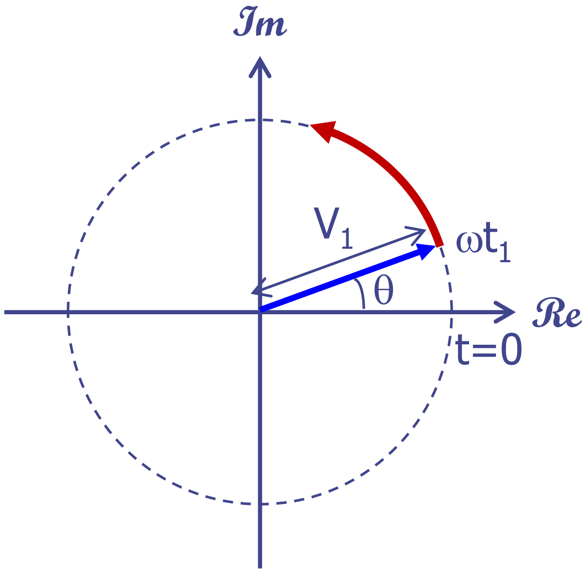
To solve the above equation, we need to assume that $v_o(t)$ takes the form of $v_o(t) = A\cos(\omega t) + B\sin(\omega t)$, and we have to deal with both $\cos(\omega t)$ and $\sin(\omega t)$. The computation is very tedious (refer to Appendix). The solution contains both a transient response and a steady-state AC response, only the latter is covered in this chapter.

Qn. Do we have a more efficient way to arrive at the answer, especially one that does not need to solve differential equations?

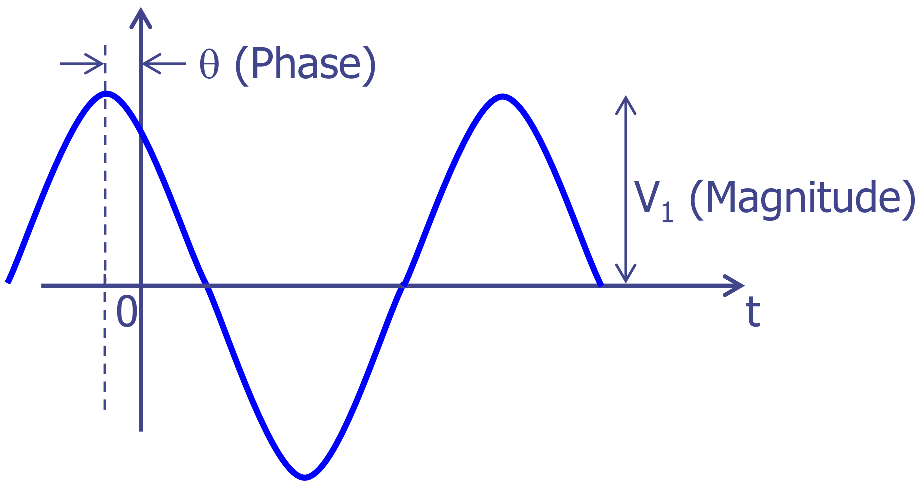
Ans. Yes, we do. But we need to use complex numbers.

Chapter 3: AC Steady-State Analysis

3.3 Phasor Analysis



Phasor Representation



Sinusoidal Waveform

3.3.1 Complex Number and Operations

The simplest example of an imaginary number is the solution to

$$x^2 + 1 = 0$$

and

$$x = \pm\sqrt{-1}$$

The number $\sqrt{-1}$ is not an integer, nor a real number, and it is regarded as an "imaginary" number in the olden days, and the name passes down to present time.

In computation, we could just treat $\sqrt{-1}$ as an authentic number, and remember that $\sqrt{-1} \times \sqrt{-1} = -1$. As this number occurs very often in mathematics, it is assigned the symbol "i". However, in electrical/electronic engineering, "i" is reserved for current, and we use the symbol "j" instead:

$$j = \sqrt{-1}$$

and

$$j \times j = -1$$

Complex Number: x + jy

Consider the quadratic equation:

$$x^2 + x + 1 = 0$$

The roots are

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

The number $-1/2 + j\sqrt{3}/2$ consists of a real part $(-1/2)$ and an imaginary part $(j\sqrt{3}/2)$, and is called a complex number. The usual symbol for complex number is "z":

$$z = x + jy$$

where x and y are real numbers, and

$$\mathcal{Re}(z) = x$$

$$\mathcal{Im}(z) = y$$

Complex Number: Addition and Subtraction

Consider two complex numbers z_1 and z_2:

$$z_1 = a + jb$$

$$z_2 = c + jd$$

Addition and Subtraction of Complex Numbers:

$$z_1 \pm z_2 = (a \pm c) + j(b \pm d)$$

Example 3-11: Given $z_1 = -3 + j7$ and $z_2 = -j12$, compute (a) $z_1 + z_2$, and (b) $z_2 - z_1$.

Soln.:

(a) $(-3 + j7) + (0 + -j12) = (-3 + 0) + j(7 - 12) = -3 - j5$

(b) $(0 + -j12) - (-3 + j7) = (0 - -3) + j(-12 - 7) = +3 - j19$

Complex Number: Multiplication

Multiplication of Complex Numbers:

$$z_1 \times z_2 = (a + jb) \times (c + jd)$$

$$= ac + jad + jbc - bd$$

$$= (ac - bd) + j\times(ad + bc)$$

Example 3-12: Compute $(-3 + j7) \times (8 + j4)$.

Soln.:

$$z_1 \times z_2 = (-3 + j7) \times (8 + j4)$$

$$= -3\times8 + j(-3\times4) + j(7\times8) - 7\times4$$

$$= -52 + j44$$

Complex Conjugate and Modulus

Before discussing division of complex number, let us introduce the complex conjugate of z, denoted as z^*, first. For

$$z = a + jb$$

$$z^* = a - jb$$

A complex number z when multiplied with its complex conjugate z^* gives a real number:

$$z \times z^* = (a + jb) \times (a - jb) = a^2 - jab + jab + b^2$$

$$= a^2 + b^2$$

The modulus of a complex number z, denoted as $|z|$, is defined as

$$|\,z\,|^2 = z \times z^* = a^2 + b^2$$

$$|\,z\,| = \sqrt{a^2 + b^2} \geq 0$$

Complex Number: Division

Division of Complex Numbers:

$$\frac{z_1}{z_2} = \frac{a+jb}{c+jd} = \frac{a+jb}{c+jd} \times \frac{c-jd}{c-jd}$$

$$= \frac{(ac+bd)+j(bc-ad)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + j \times \frac{bc-ad}{c^2+d^2}$$

Example 3-13: Compute (–3 + j7) / (8 + j4).

Soln.:

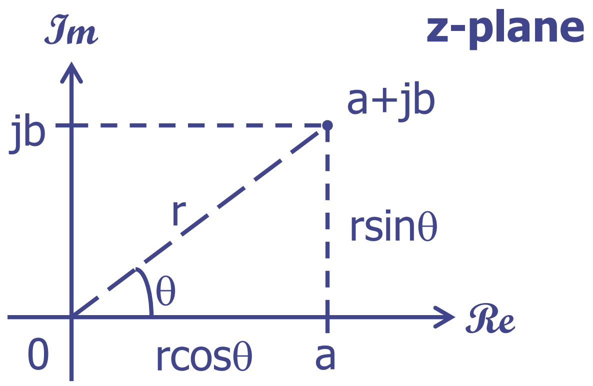
$$\frac{-3+j7}{8+j4} = \frac{-3+j7}{8+j4} \times \frac{8-j4}{8-j4}$$

$$= \frac{(-24+28)+j(56+12)}{8^2+4^2} = \frac{4+j68}{80}$$

$$= 0.05 + j0.85$$

Complex Plane, Rectangular and Polar Forms

Complex numbers are conveniently drawn on the complex plane (z-plane). The x-axis is the real axis ($\mathscr{Re}$-axis), and the y-axis is the imaginary axis ($\mathscr{Im}$-axis).



$$r = \sqrt{a^2 + b^2} \geq 0$$

$$\tan\theta = \frac{b}{a}$$

On the z-plane, complex numbers can be expressed as

	z	$= a + jb$	rectangular form
or	z	$= r\cos\theta + jr\sin\theta$	
with	r	$= \lvert z \rvert = \sqrt{a^2 + b^2} \geq 0$	
	θ	$= \tan^{-1}(b / a)$	
	$\mathscr{Re}(z)$	$= a = r\cos\theta$	
	$\mathscr{Im}(z)$	$= b = r\sin\theta$	

3.3.2 Euler's Equation of Complex Exponentials

From study of calculus (using Taylor and Maclaurin series expansions), we learn:

Leonhard Euler
1707 – 1783

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Euler

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Define

$$e^z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \cdots$$

For $z = j\theta$

$$e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + j\sin\theta$$

https://en.wikipedia.org/wiki/Leonhard_Euler

Euler's Equation of Complex Exponentials (cont.)

Euler's equation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

The complex exponential function $e^{j\theta}$ can be viewed as more fundamental than the $\sin\theta$ and $\cos\theta$ functions, which are both derivable from $e^{j\theta}$:

$$\cos\theta = \mathcal{Re}\left(e^{j\theta}\right), \qquad \sin\theta = \mathcal{Im}\left(e^{j\theta}\right)$$

In particular

$$e^{j\pi} + 1 = 0$$

The Most Beautiful Formula in Mathematics

Compared with Einstein's equation

$$E = mc^2$$

The Most Famous Equation in Physics

Complex Numbers on Unit Circle

Example 3-14: Common complex numbers on the unit circle.

$$e^{j0} = e^0 = 1$$

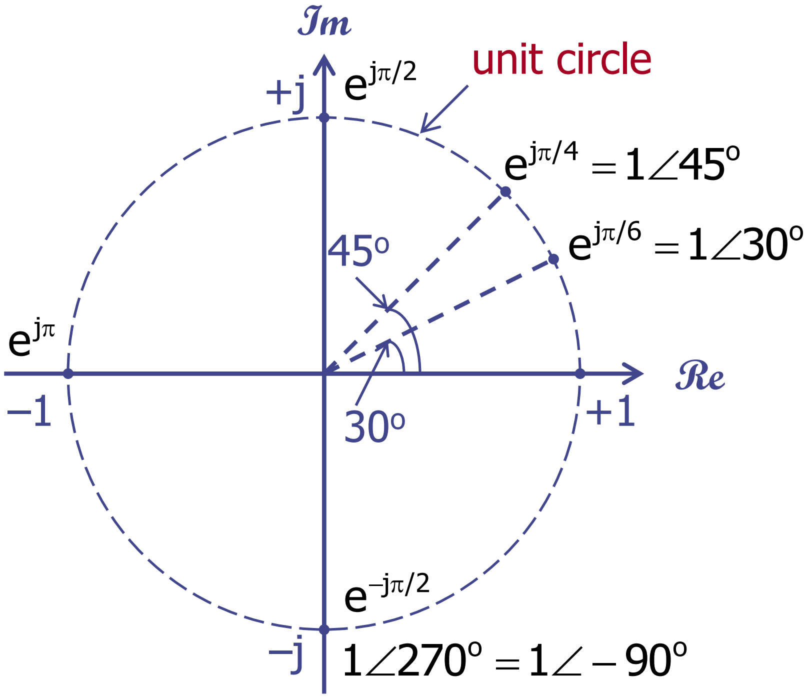
$$e^{j\pi/6} = \cos 30^o + j\sin 30^o = \frac{2}{\sqrt{3}} + j\frac{1}{\sqrt{3}}$$

$$e^{j\pi/4} = \cos 45^o + j\sin 45^o = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$e^{j\pi/2} = \cos 90^o + j\sin 90^o = j$$

$$e^{j\pi} = \cos 180^o + j\sin 180^o = -1$$

$$e^{j3\pi/2} = \cos 270^o + j\sin 270^o = -j$$



$$\angle\theta \equiv e^{j\theta} = \cos\theta + j\sin\theta$$

Polar Form

A complex number z in polar form can be written as

$$z = re^{j\theta} \qquad r \geq 0$$

In electrical/electronic engineering, the polar form is usually written as

$$z = r\angle\theta \qquad r \geq 0 \text{ and } \angle\theta \equiv e^{j\theta}$$

The modulus r is better known as the magnitude of the complex number, and θ as the phase.

Multiplication and Division in Polar Form

Multiplication in polar form:

$$z_3 = z_1 \times z_2 = r_1 e^{j\theta_1} \times r_2 e^{j\theta_2}$$

$$= r_1 r_2 e^{j(\theta_1+\theta_2)}$$

or $$z_3 = r_1\angle\theta_1 \times r_2\angle\theta_2$$

$$= r_1 r_2 \angle(\theta_1 + \theta_2)$$

Division in polar form:

$$z_3 = \frac{z_1}{z_2} = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}$$

$$= (r_1 / r_2) e^{j(\theta_1-\theta_2)}$$

or $$z_3 = \frac{r_1\angle\theta_1}{r_2\angle\theta_2}$$

$$= (r_1 / r_2)\angle(\theta_1 - \theta_2)$$

Example 3-15

Example 3-15: Given $z_1 = 2 + j6$ and $z_2 = -4 - j3$. (a) Write the polar form of z_1 and z_2; (b) compute $z_1 z_2$ in polar form and convert the answer into rectangular form; and (c) compute z_1/z_2 in polar form and convert the answer into rectangular form.

Soln.:

(a)

$$r_1 = \sqrt{2^2 + 6^2} = 6.325$$

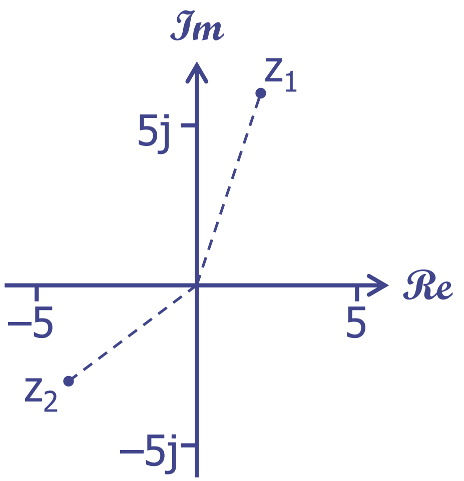
$$\theta_1 = \tan^{-1}\frac{6}{2} = 71.57^\circ$$

$$z_1 = 6.325\angle 71.57^\circ$$

$$r_2 = \sqrt{4^2 + 3^2} = 5$$

$$\theta_2 = \tan^{-1}\frac{-3}{-4} = 216.87^\circ$$

$$z_2 = 5\angle 216.87^\circ \quad (\text{NOT } 5\angle 36.87^\circ)$$



Special care is needed in converting the complex number in quadrants 2, 3, and 4.

Example 3-15 (cont.)

(b) $z_1 \times z_2$

$$= (6.325 \times 5)\angle(71.57^o + 216.87^o)$$
$$= 31.63\angle 288.4^o = 31.6\angle -71.6^o$$
$$= 31.63 \times (\cos(288.4^o) + j\sin(288.4^o))$$
$$= 9.98 - j \times 30.0$$

(c) $\frac{z_1}{z_2}$

$$= \frac{6.325}{5}\angle(71.57^o - 216.87^o)$$
$$= 1.265\angle -145.3^o = 1.26\angle -145.3^o$$
$$= 1.265 \times (\cos(-145.3^o) + j\sin(-145.3^o))$$
$$= -1.04 - j \times 0.72$$

Note that usually answers with 3 significant digits are good enough, and we should use 4-digit accuracy for computing intermediate results.

3.3.3 Complex Sinusoid as Excitation

Qn. What is the connection between complex number and circuit analysis?

Ans. The ingenuity is to replace the real sinusoidal source $V_s\cos(\omega t)$ by the complex sinusoidal source $V_s e^{j\omega t}$.

Let complex voltages and currents be expressed as $\{v(t)\}^{c}$ and $\{i(t)\}^{c}$:

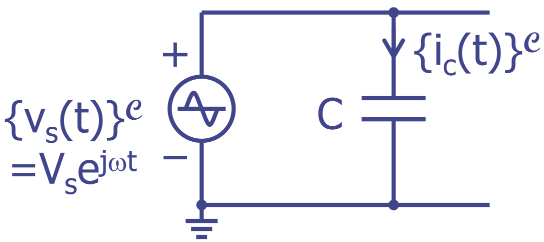
$$v_s(t) = V_s\cos(\omega t) \rightarrow \{v_s(t)\}^{c} = V_s e^{j\omega t}$$

Use $\{v_s(t)\}^{c}$ to compute the capacitor current:

$$\{i_c(t)\}^{c} = C\frac{d\{v_s(t)\}^{c}}{dt} = C\frac{d}{dt}\left(V_s e^{j\omega t}\right)$$

$$= j\omega C V_s e^{j\omega t}$$

$$= j\omega C \cdot \{v_s(t)\}^{c}$$



$\text{Re}(\{i_c(t)\}^{\mathcal{C}})=i_c(t)$ and $1/j\omega C$ Resembles Resistance

Two observations are in place.

(1) The real part of $\{i_c(t)\}^{\mathcal{C}}$ is the correct response to the original real source:

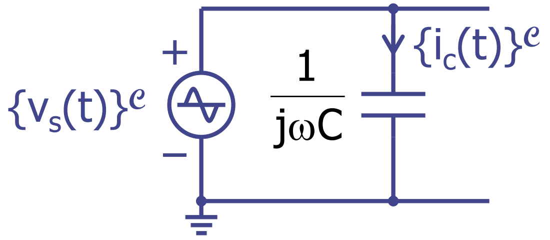
$$\{i_c(t)\}^{\mathcal{C}} = j\omega C V_s(\cos(\omega t) + j\sin(\omega t))$$

$$= -\omega C V_s \sin(\omega t) + j\omega C V_s \cos(\omega t)$$

$$\text{Re}(\{i_c(t)\}^{\mathcal{C}}) = i_c(t) = -\omega C V_s \sin(\omega t)$$

(2) The ratio of complex voltage to complex current resembles a "resistance":

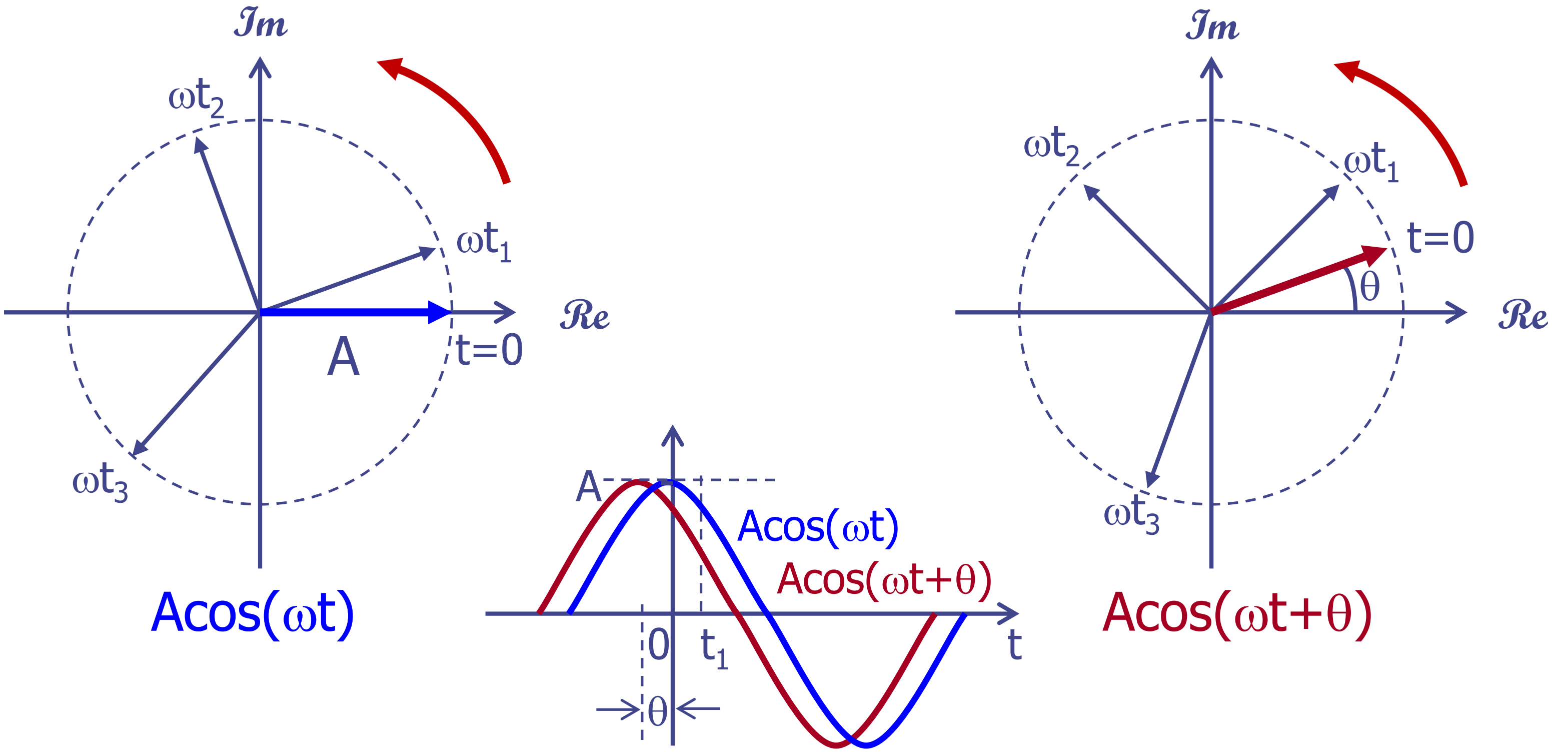
$$\frac{\{v_s(t)\}^{\mathcal{C}}}{\{i_c(t)\}^{\mathcal{C}}} = \frac{1}{j\omega C}$$



The ultimate simplification is to use phasors.

Rotating Vector

Consider $x(t) = A\cos(\omega t)$, and let us represent it as a rotating vector sweeping on a circle.



$x(t)$ can be thought to rotate on the complex plane, with $\{x(t)\}^{\mathcal{C}} = Ae^{j\omega t}$. Each $\{x(t_i)\}^{\mathcal{C}}$ is a point on the circle with radius A, and the real part is $x(t_i)$.

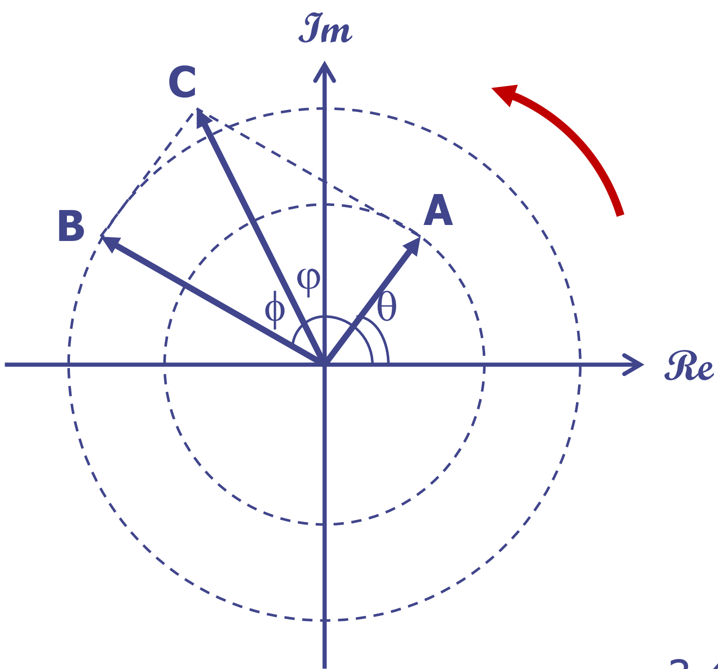
3.3.4 Phasors

If x(t) has an initial phase, say, $x(t) = A\cos(\omega t+\theta)$, it can be represented by the right figure of the previous page. Note that all subsequent positions of the vector are known once the initial phase is determined, as the angular frequency ω is given.

We can represent two functions, $a(t) = A\cos(\omega t+\theta)$ and $b(t) = B\cos(\omega t+\phi)$, having the same ω on the same diagram, specifying only the initial positions.

$$\{a(t)\}^{\mathcal{C}} = Ae^{j\theta}e^{j\omega t} = \mathbf{A}e^{j\omega t}$$

$$\{b(t)\}^{\mathcal{C}} = Be^{j\phi}e^{j\omega t} = \mathbf{B}e^{j\omega t}$$



We then define the phasors of a(t) and b(t) as

$$\mathbf{A} = Ae^{j\theta} = A\angle\theta$$

$$\mathbf{B} = Be^{j\phi} = B\angle\phi$$

Operations of Phasors

It is clear that the term phasor refers to a rotating vector specified by its initial phase. For the time being, we use boldface letters to indicate phasors. Now, real signals a(t) and b(t) are represented as phasors **A** and **B** that are complex numbers:

$$\mathbf{A} = A\angle\theta = a_1 + ja_2$$

$$\mathbf{B} = B\angle\phi = b_1 + jb_2$$

Operations of phasors follow that of complex numbers. For addition:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = C\angle\varphi = c_1 + jc_2$$

$$= (a_1 + b_1) + j(a_2 + b_2)$$

with

$$C = \sqrt{c_1{}^2 + c_2{}^2} = \sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2}$$

$$\varphi = \tan^{-1}\frac{c_2}{c_1} = \tan^{-1}\frac{a_2 + b_2}{a_1 + b_1}$$

Example 3-16

Example 3-16:

Let $a(t) = 10\times\cos(30t+50^o)$

$b(t) = 5\times\cos(30t+125^o)$

Find $c(t) = a(t) + b(t)$ using phasor method.

Evaluate c(t) at t = 0.1 s.

Soln.:

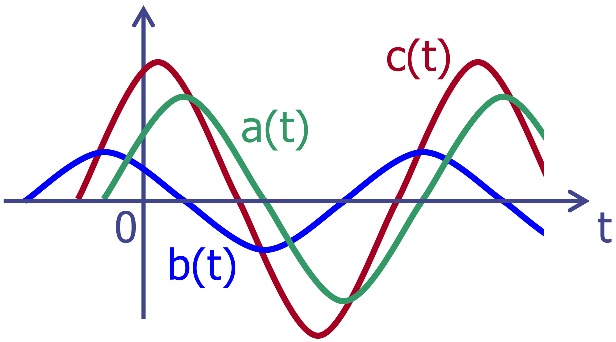
$$a(t) = 10\times\cos(30t+50^o) \Rightarrow \mathbf{A} = 10\angle 50^o = 6.428 + j7.660$$

$$b(t) = 5\times\cos(30t+125^o) \Rightarrow \mathbf{B} = 5\angle 125^o = -2.868 + j4.096$$

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = 3.560 + j11.756 = 12.28\angle 73.15^o$$

$$\Rightarrow c(t) = 12.28\times\cos(30t+73.15^o)$$

$$c(0.1) = 12.28\times\cos(3 \text{ rad} + 73.15^o)$$
$$= 12.28\times\cos(3 + \pi\times 73.15^o/180^o)$$
$$= 12.28\times\cos(4.277) = -5.18$$



Examples 3-17, 3-18

Example 3-17: Let a(t) = 10×cos(30t+50°) and d(t) = 5×sin(30t+125°), find e(t) = a(t) + d(t).

Soln.:

Note that sin(θ) lags cos(θ) by 90°. By writing sin(θ) = cos(θ – 90°), we can express d(t) = 5×cos(30t+35°). The phasors of a(t) and d(t) are now compatible. Hence,

$$\mathbf{D} = 5\angle 35^{o} = 4.096 + j2.868$$

$$\mathbf{E} = \mathbf{A} + \mathbf{D} = 10.524 + j10.528 = 14.9\angle 45.0^{o}$$

$$\Rightarrow \quad e(t) = 14.9\cos(30t + 45.0^{o})$$

Here we can't use sin(θ) = cos(90° – θ) as d(t) would be 5×cos(–30t –35°) then, resulting in a vector rotating in the opposite direction.

Example 3-18: Let a(t) = 10×sin(30t+50°) and f(t) = 5×sin(45t+125°), find g(t) = a(t) + f(t).

Soln.:

a(t) and f(t) have different frequencies and cannot be added together using phasors, and g(t) remains

g(t) = 10×sin(30t+50°) + 5×sin(45t+125°)

Example 3-19

Example 3-19: Let $m(t) = 10\times\sin(30t+50^o)$ and $n(t) = 5\times\sin(30t+125^o)$, find $k(t) = m(t) + n(t)$.

Soln.:

Method 1:

$m(t) = 10\times\sin(30t+50^o) = 10\times\cos(30t-40^o) \Rightarrow \mathbf{M} = 10\angle-40^o$

$n(t) = 5\times\sin(30t+125^o) = 5\times\cos(30t+35^o) \Rightarrow \mathbf{N} = 5\angle35^o$

$\mathbf{K} = \mathbf{M} + \mathbf{N} = 10\angle-40^o + 5\angle35^o = 7.660 - j6.428 + 4.096 + j2.868$

$= 11.756 - j3.560 = 12.3\angle-16.8^o$

$\Rightarrow k(t) = 12.3\cos(30t-16.8^o) = 12.3\sin(30t+73.2^o)$

Method 2:

Note that m(t) and n(t) are just a(t) and b(t) replaced with the sine function (or offset by -90^o). We may simply add **A** and **B** to obtain **C** as in Example 3-16, but remember to put the answer with reference to sine, that is,

$$k(t) = 12.3\sin(30t + 73.2^o)$$

$d/dt \rightarrow j\omega$ and $\int d\lambda \rightarrow 1/j\omega$

For

$$\{a(t)\}^{e} = \mathbf{A}e^{j\omega t}$$

we have

$$\frac{d\{a(t)\}^{e}}{dt} = \frac{d}{dt}\left(\mathbf{A}e^{j\omega t}\right) = j\omega\left(\mathbf{A}e^{j\omega t}\right)$$

which looks as if the differential operator d/dt is replaced by $j\omega$:

$$\frac{d}{dt} \rightarrow j\omega$$

Likewise, the integral operator $\int d\lambda$ is replaced by $1/j\omega$:

$$\int^{t} d\lambda \rightarrow \frac{1}{j\omega}$$

By using phasors, differential equations are turned to algebraic equations that are much easier to solve.

3.3.5 Impedance and Admittance

Differentiation and integration are due to the presence of capacitors and inductors. Let us reconsider driving a capacitor C with a voltage source $v_s(t) = V_s\cos(\omega t+\theta) = v_c(t)$, and

$$\{v_c(t)\}^{\mathcal{e}} = \mathbf{V_c} e^{j\omega t}$$

Now, the capacitor current (the response) due to the forced oscillation must have the same frequency, that is,

$$\{i_c(t)\}^{\mathcal{e}} = \mathbf{I_c} e^{j\omega t}$$

In fact,

$$\{i_c(t)\}^{\mathcal{e}} = C\frac{d}{dt}\left(\mathbf{V_c} e^{j\omega t}\right) = j\omega C \mathbf{V_c} e^{j\omega t}$$

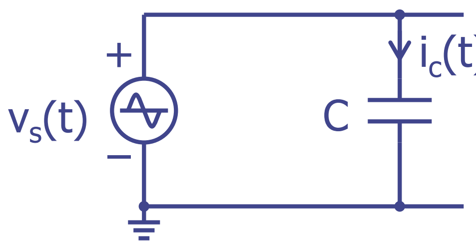
giving

$$\frac{\mathbf{V_c}}{\mathbf{I_c}} = z_c(j\omega) = \frac{1}{j\omega C}$$

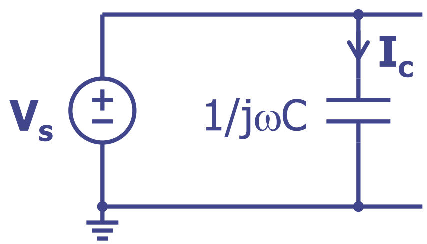
The ratio $\mathbf{V_c}/\mathbf{I_c} = z_c(j\omega)$ is NOT a phasor, but is the impedance (resembling resistance) of the capacitor C.

Admittance of Capacitor

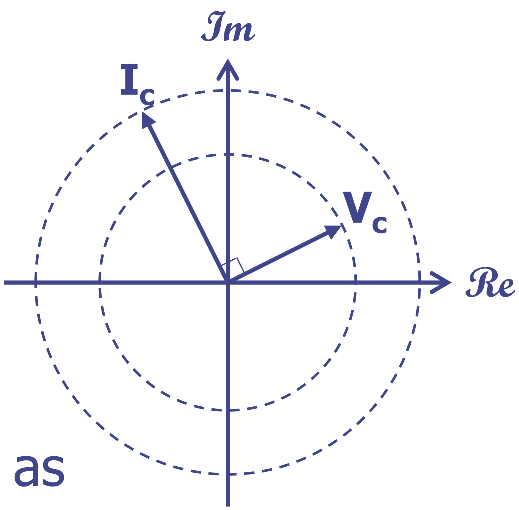
Time-domain analysis and phasor analysis can be summarized as shown in the figures below, and from the phasor diagram, $\mathbf{I_c}$ leads $\mathbf{V_c}$ by 90^o is immediately observed.



Time-domain analysis



Phasor analysis



Rewrite the I-V characteristic of the capacitor as

$$\frac{\mathbf{I_c}}{\mathbf{V_c}} = y_c(j\omega) = j\omega C$$

and $y_c(j\omega)$ is the admittance (resembling conductance) of C. The capacitor current leads the source voltage by 90^o is easily observed by writing

$$\mathbf{I_c} = j\omega C\mathbf{V_c}$$

Impedance and Admittance of Inductor

Consider driving an inductor L with $v_s(t) = V_s\cos(\omega t+\theta) = v_\ell(t)$:

$$\{v_\ell(t)\}^{\mathcal{C}} = \mathbf{V}_\ell e^{j\omega t}$$

The inductor current will have the same frequency as $v_s(t)$:

$$\{i_\ell(t)\}^{\mathcal{C}} = \mathbf{I}_\ell e^{j\omega t}$$

and

$$\{i_\ell(t)\}^{\mathcal{C}} = \frac{1}{L}\int^{t} \mathbf{V}_\ell e^{j\omega\lambda} d\lambda = \frac{1}{j\omega L}\mathbf{V}_\ell e^{j\omega t}$$

$$\Rightarrow \quad \frac{\mathbf{V}_\ell}{\mathbf{I}_\ell} = z_\ell(j\omega) = j\omega L$$

The impedance of the inductor is $j\omega L$, and the admittance is

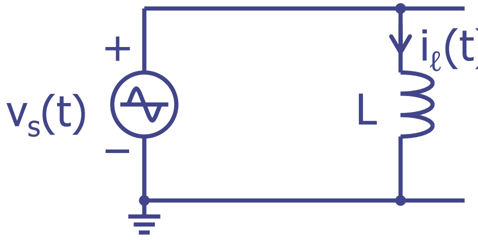
$$\frac{\mathbf{I}_\ell}{\mathbf{V}_\ell} = y_\ell(j\omega) = \frac{1}{j\omega L}$$

$$\Rightarrow \quad \mathbf{I}_\ell = \frac{1}{j\omega L}\mathbf{V}_\ell = \frac{-j}{\omega L}\mathbf{V}_\ell$$

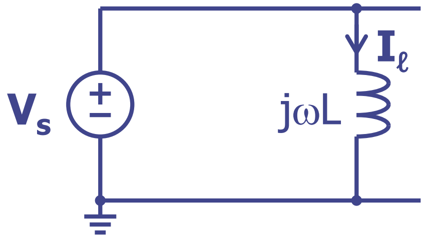
The inductor current lags the inductor voltage by 90^o.

Phasor Analysis of Inductor

Time-domain analysis and phasor analysis of the inductor is summarized in the figures below.

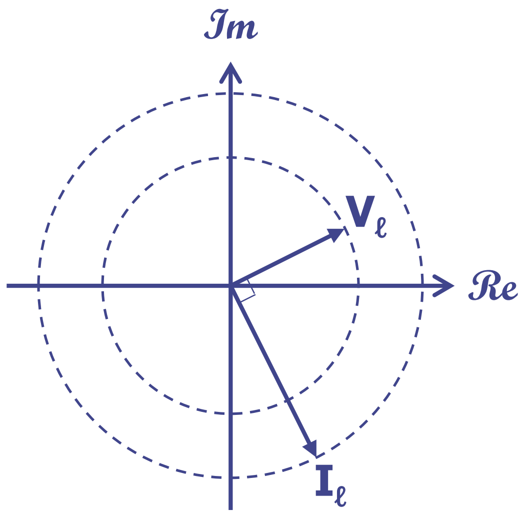


Time-domain analysis



Phasor analysis

From the phasor diagram, $\mathbf{I}_\ell$ lags $\mathbf{V}_\ell$ by 90^o is immediately observed:



Impedance and Admittance

Impedance and admittance are equivalent to resistance and conductance in resistive networks, and they are not phasors. They are collectively known as immittance.
In general, immittance are complex, instead of just being imaginary as in the case of a capacitor or an inductor.

Impedance: $Z = R + jX$ R is resistance, X is reactance
Admittance: $Y = G + jB$ G is conductance, B is susceptance

Reactance of Capacitor and Inductor (Unit is Ω)

Capacitor: $Z_c = \frac{1}{j\omega C} = +jX_c \Rightarrow X_c = -\frac{1}{\omega C}$

Inductor: $Z_\ell = j\omega L = +jX_\ell \Rightarrow X_\ell = +\omega L$

3.3.6 Phasor Analysis

Procedure of Phasor Analysis

(1) Express all time-dependent terms as cosine functions and then convert them to their phasor equivalents, for example, $V_s\cos(\omega t+\theta) \rightarrow \mathbf{V_s}$ ($=V_s\angle\theta$). Voltages and currents are no longer functions of time, but are phasors. Moreover, the symbol of sinusoidal source can be replaced by the symbol of DC source.

(2) Substitute the capacitor C with $1/j\omega C$ and the inductor L with $j\omega L$ as the impedance, while the resistor R remains unchanged. A phasor circuit is then obtained.

(3) Treat the phasor circuit as a resistive circuit, and apply the same DC analysis methods, e.g., KCL, KVL, superposition, to solve for the unknown parameters in terms of phasors.

(4) If time-domain solution is needed, convert the parameters in phasors back to their time-domain equivalents.

Common Practice in Phasor Analysis

As phasor analysis is the principal method in analyzing sinusoidal steady state, engineers want to be more efficient in notations.

(1) Very often, phasors are not written in boldface. For convenience, we may just use the time-domain symbol as the phasor.

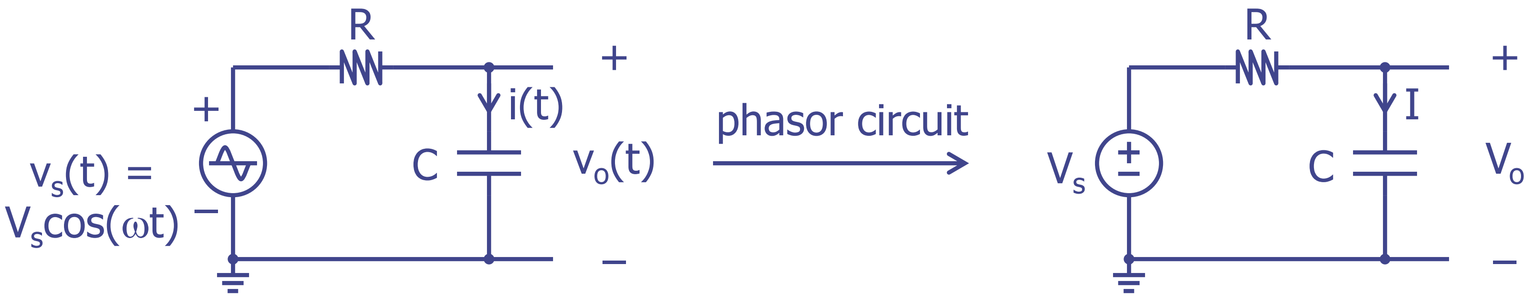
For example, for $v_s(t) = V_s\cos(\omega t+\theta)$, the phasor should be $\mathbf{V}_s$, but we may use $V_s\angle\theta$, or simply, V_s (if $\theta=0$) to represent the phasor instead.

(2) Capacitors and inductors are not represented by $1/j\omega C$ and $j\omega L$, but are still represented by C and L, but understood to use $1/j\omega C$ and $j\omega L$ in actual computation.

Note: Real world signals must be expressed in cosine functions and not sine functions before converting to phasors. We get back to the real world by taking the real part, i.e., the cosine portion.

Phasor Analysis of RC Circuit

Equipped with phasor analysis, let us reconsider the RC circuit in p. 3-22.



In the phasor domain, R and C (R and $1/j\omega C$) form a voltage divider, and

$$V_o = \frac{1/j\omega C}{R + 1/j\omega C} V_s$$

$$\Rightarrow \quad V_o = \frac{1}{1 + j\omega CR} V_s$$

In symbolic form, the above is already the answer!

Phasor Analysis of RC Circuit (cont.)

If numerical values are given for V_s, f, R and C, then we could obtain the numerical value of V_o. Hence, we need to compute

$$V_o = \frac{1}{1 + j\omega CR} V_s$$
$$= \frac{1}{\sqrt{1 + \omega^2 C^2 R^2} \angle \phi} V_s$$
$$= \frac{V_s}{\sqrt{1 + \omega^2 C^2 R^2}} \angle(-\phi)$$

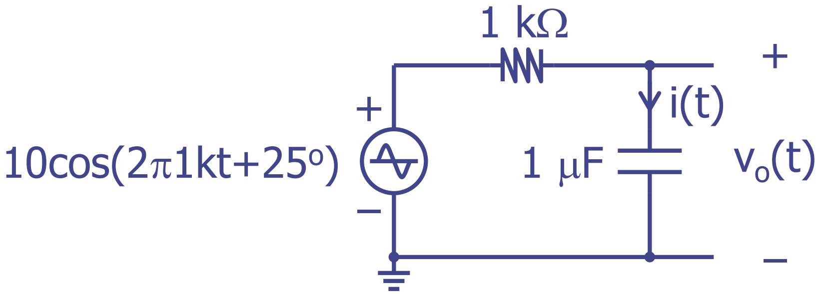
where $$\phi = \tan^{-1}(\omega CR)$$

If time-domain result is needed, then convert the V_o phasor back to its time equivalent:

$$v_o(t) = \frac{V_s}{\sqrt{1 + \omega^2 C^2 R^2}} \cos(\omega t - \phi)$$

Example 3-20

Example 3-20: Solve for $v_o(t)$ of the RC circuit below.



Soln.:
Although the source has an initial phase of 25°, we may still use 10 as the source (instead of $10\angle 25^o$), and just remember to reference to +25° when we put down the final answer.

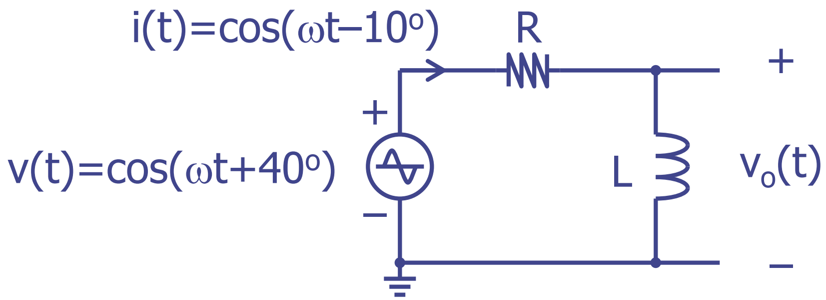
$$V_o = \frac{1}{1 + j2\pi \times 1k \times 1\mu \times 1k} \times 10 = \frac{10}{1 + j6.283}$$

$$= 1.57\angle -81^o$$

$$v_o(t) = 1.57\cos(2\pi 1kt - 56^o)$$

Example 3-21

Example 3-21: Solve for R and L of the circuit below, given f=100 Hz.



Soln.:

Clearly, when written in phasor notation we have

$$\frac{V}{I} = R + j\omega L = \frac{\angle 40^o}{\angle -10^o} = \angle 50^o = \cos 50^o + j\sin 50^o$$

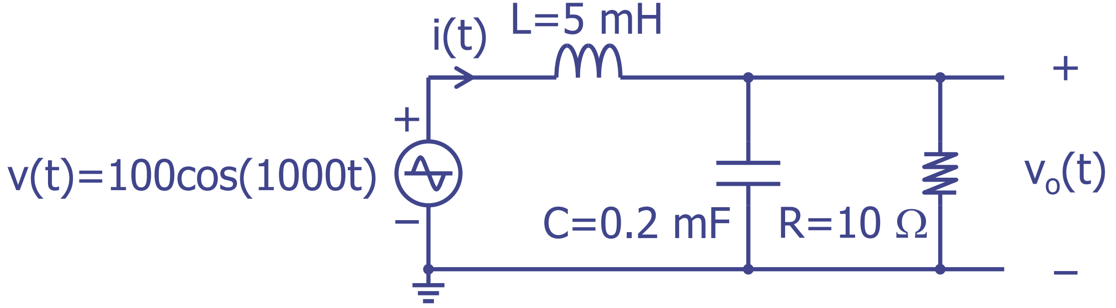
$$\Rightarrow \quad R = \cos 50^o = 0.643\Omega$$

$$\omega L = \sin 50^o = 0.766$$

$$L = \frac{0.766}{2\pi \times 100} = 1.22\text{mH}$$

Example 3-22

Example 3-22: Compute the current i(t) and the output voltage $v_o(t)$ of the following circuit.



Soln.:

One may choose to compute the impedance of the L and C first:

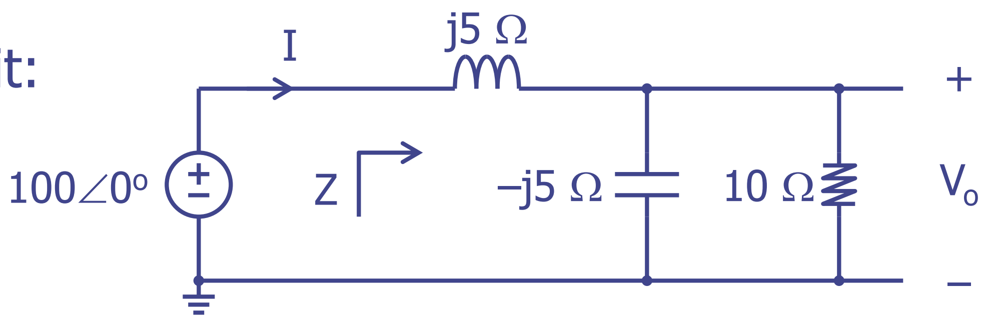
$$v(t) \rightarrow 100\angle 0^{o}$$

$$j\omega L = j1000 \times 5m = j5\Omega$$

$$\frac{1}{j\omega C} = \frac{1}{j1000 \times 0.2m} = \frac{1}{j0.2} = -j5\Omega$$

Example 3-22 (cont.)

Phasor circuit:



$$Z = j5 + (-j5 \,||\, 10) = j5 + \frac{-j5 \times 10}{10 - j5} = j5 - \frac{j10}{2-j}\cdot\frac{2+j}{2+j}$$

$$= j5 - \frac{-10 + j20}{4+1} = j5 + 2 - j4 = 2 + j = \sqrt{5}\angle 26.57^o$$

$$I = \frac{V}{Z} = \frac{100}{\sqrt{5}\angle 26.57^o} = 20\sqrt{5}\angle -26.57^o = 44.72\angle -26.57^o$$

$$i(t) = 44.7\cos(1000t - 26.6^o) \ \ (A)$$

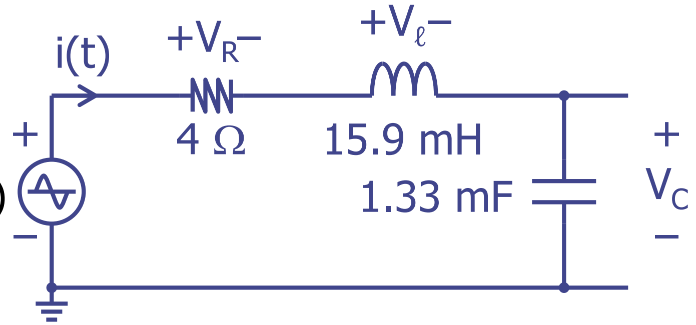
$$V_o = I \times (-j5 \,||\, 10) = I \times (2 - j4)$$

$$= 44.72\angle -26.57^o \times 4.472\angle -63.43^o = 200\angle -90^o$$

$$v_o(t) = 200\cos(1000t - 90^o) \ \ (V)$$

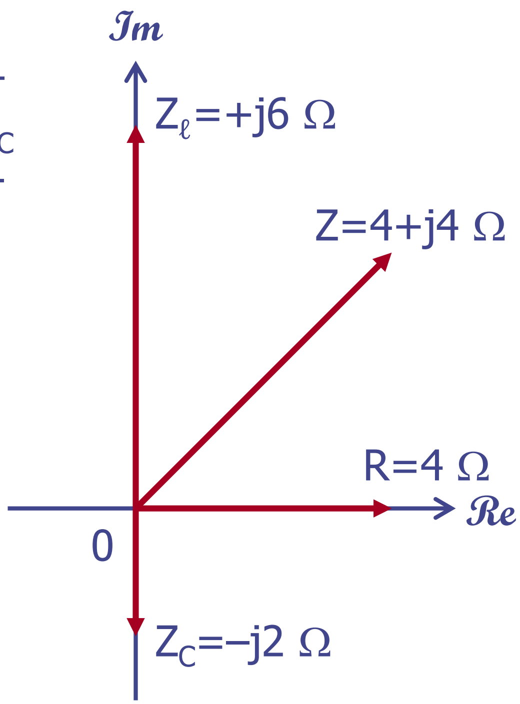
Example 3-23

Example 3-23: (1) Find and plot all impedances on one z-plane; and (2) find and plot the current i(t) and all voltages on a second z-plane.



$v_s(t) = 12\sqrt{2}\cos(377t + 90^o)$

Impedance Diagram



Soln.:

(1)

$$Z_\ell = j\omega L = j377 \times 15.9m = +j6$$

$$Z_C = \frac{1}{j\omega C} = \frac{1}{j377 \times 1.33m} = -j2$$

$$Z = R + j\omega L + \frac{1}{j\omega C} = 4 + j4$$

Example 3-23 (cont.)

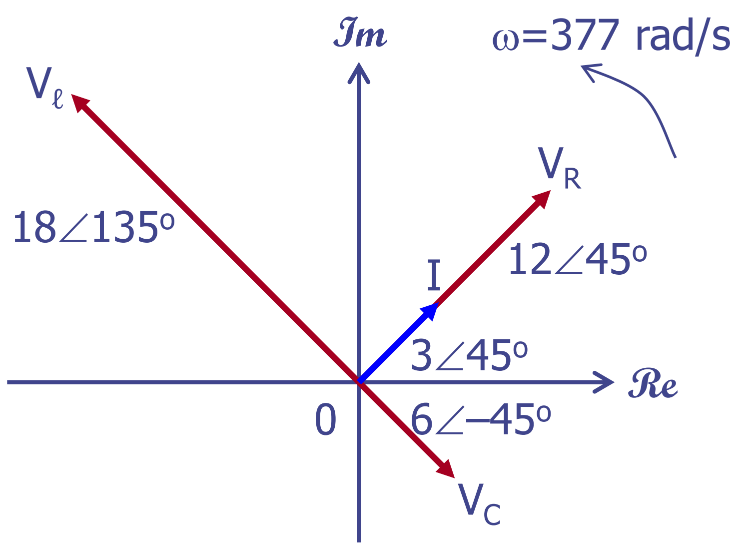
(2)

$$I = \frac{V_s}{Z} = \frac{12\sqrt{2}\angle 90^o}{4+j4} = \frac{12\sqrt{2}\angle 90^o}{4\sqrt{2}\angle 45^o} = 3\angle 45^o \ \ (A)$$

$$V_R = I \times R = 3\angle 45^o \times 4 = 12\angle 45^o \ \ (V)$$

$$V_\ell = I \times j\omega L = 3\angle 45^o \times j6 = 18\angle 135^o \ \ (V)$$

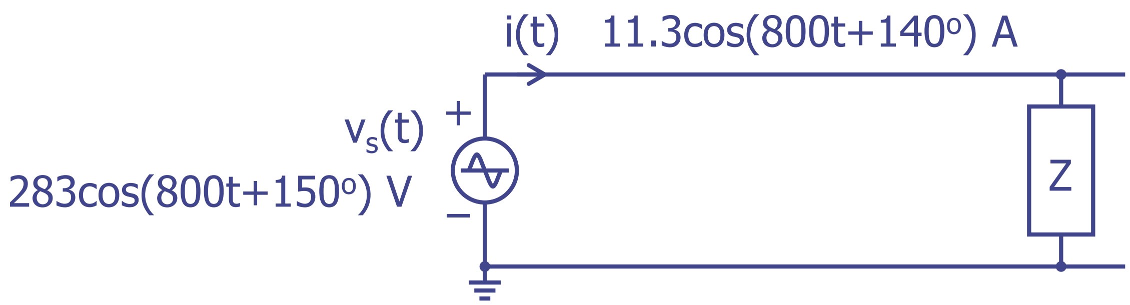
$$V_C = I / j\omega C = 3\angle 45^o \times -j2 = 6\angle -45^o \ \ (V)$$



Phasor Diagram

Example 3-24

Example 3-24: Find the unknown elements of Z if it consists of two components in series.



Soln.

(1) Transform $v_s(t)$ and i(t) into phasors V_s and I:

$$V_s = 283\angle 150^o$$

$$I = 11.3\angle 140^o$$

(2) Find z:

$$Z = \frac{V_s}{I} = \frac{283\angle 150^o}{11.3\angle 140^o} = 25.04\angle 10^o$$

$$= 24.66 + j4.348\ \Omega$$

Example 3-24 (cont.)

(3) Because z has real and imaginary parts, and the two components are in series, and

R and L gives $R + j\omega L$

R and C gives $R + 1/j\omega C = R - j/\omega C$

Hence,

$$Z = R + j\omega L$$

(4) Equate

$$24.66 + j4.348 = R + j \times 800 \times L$$

we have

$$R = 24.7\Omega$$

$$L = \frac{4.348}{800} = 5.44\text{mH}$$

Example 3-25

Example 3-25: Using the same figure as in Example 3-24, find Z if it consists of two components in parallel.

Soln.

For the two components to be in parallel:

R and L gives $R \,||\, j\omega L = \dfrac{R \times j\omega L}{R + j\omega L} = \dfrac{\omega LR(\omega L + jR)}{R^2 + \omega^2 L^2}$

Solving this problem using impedance is quite tedious, but rather straightforward if admittance is used.

$$Y = \frac{I}{V_s} = 0.03994\angle -10^o = 0.03933 - j0.006936$$

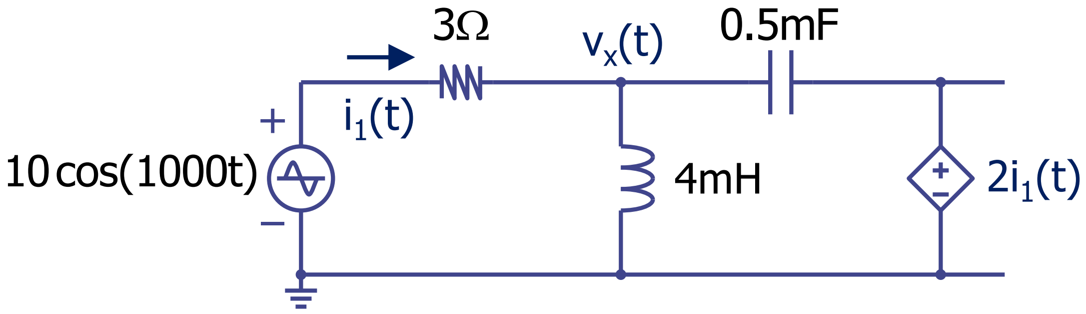
$$= \frac{1}{R} + \frac{1}{j\omega L}$$

$$\Rightarrow \quad R = 25.4\Omega$$

$$L = \frac{1}{800 \times 0.006936} = 180\text{mH}$$

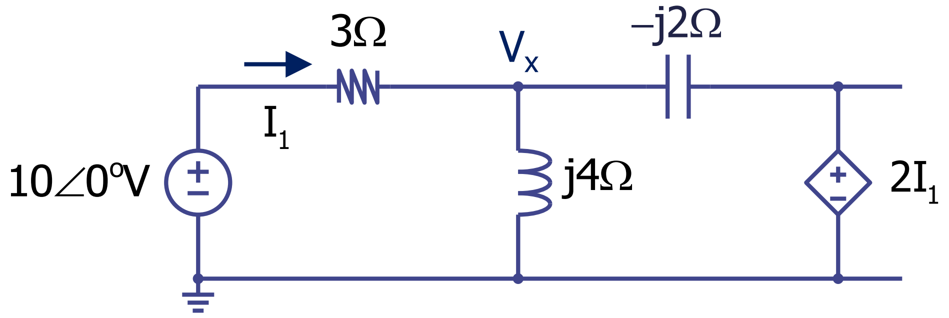
Example 3-26

Example 3-26 (Nodal Analysis): Find $v_x(t)$ and $i_1(t)$.



Soln.:

$$Z_\ell = j\omega L = j\times1000\times4m = j4\ \Omega$$
$$Z_c = 1/j\omega C = -j/(1000\times0.5m) = -j2\ \Omega$$



Example 3-26 (cont.)

KVL along the left branch yields

$$V_x = 10 - 3I_1 \qquad (1)$$

Apply KCL to node V_x :

$$-I_1 + \frac{V_x}{\mathrm{j}4} + \frac{V_x - 2I_1}{-\mathrm{j}2} = 0$$

$$-I_1 + \frac{10 - 3I_1}{\mathrm{j}4} + \frac{10 - 5I_1}{-\mathrm{j}2} = 0$$

$$-\mathrm{j}4I_1 + 10 - 3I_1 - 20 + 10I_1 = 0$$

$$I_1 = \frac{10}{7 - \mathrm{j}4} = \frac{10}{65}(7 + \mathrm{j}4) = 1.0770 + \mathrm{j}0.6154 = 1.24\angle 29.7°$$

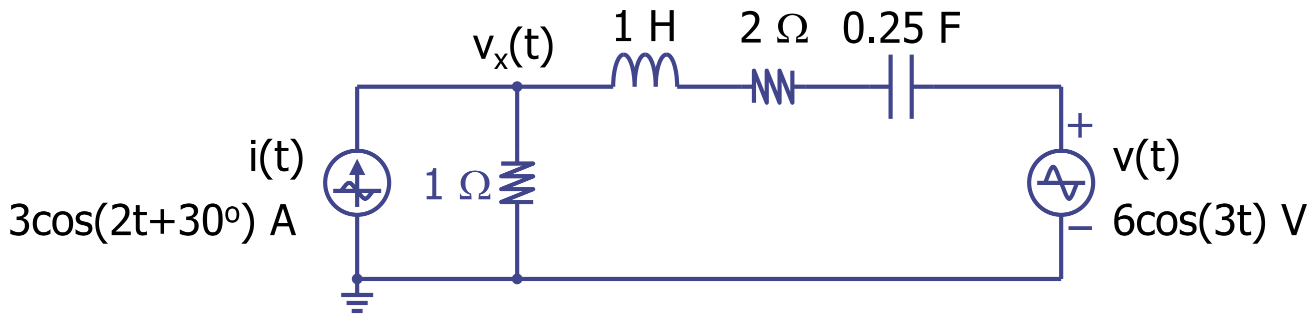
From (1) $\quad V_x = 10 - 3(1.0770 + \mathrm{j}0.6154) = 7.02\angle(-15.3)°$

$\Rightarrow \quad i_1(t) = 1.24\cos(1000t + 29.7°)$ A

$v_x(t) = 7.02\cos(1000t - 15.3°)$ V

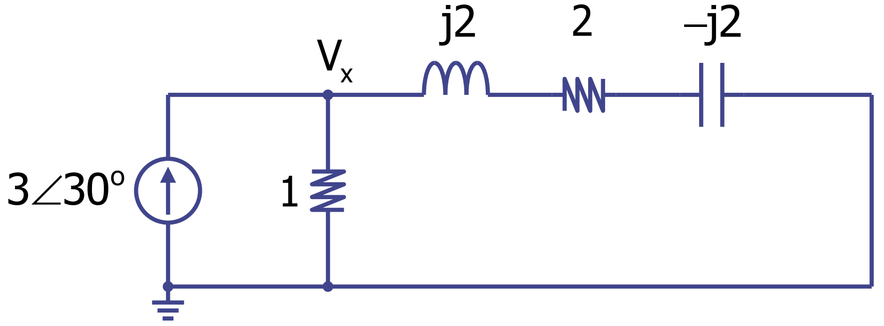
Example 3-27

Example 3-27 (Superposition): Find $v_x(t)$ of the circuit below.



Soln. Note that the two sources have different frequencies, and we may use superposition in combination of phasor analysis to solve this problem.

(1) Find V_x due to I first ($\omega = 2$):



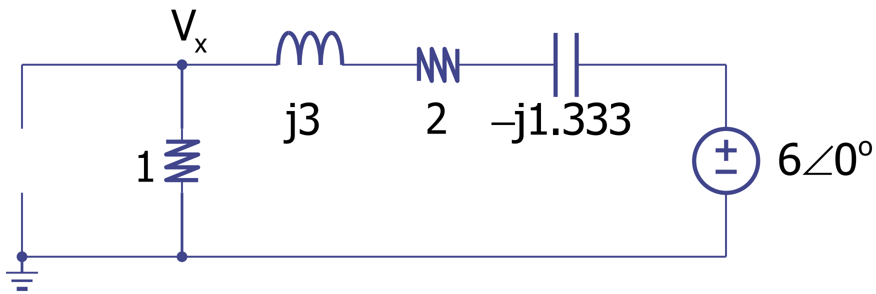
Example 3-27 (cont.)

The impedance of the inductor cancels that of the capacitor, and

$$V_x = 3\angle 30^o \times (1 \,||\, 2) = 2\angle 30^o$$

$$\Rightarrow \quad v_x(t)\big|_{i(t)} = 2\cos(2t + 30^o)$$

(2) Next, find V_x due to V (ω = 3):



$$V_x = \frac{1}{1 + 2 + j3 - j1.333} 6\angle 0^o = \frac{6\angle 0^o}{3 + j1.666} = \frac{6\angle 0^o}{3.432\angle 29.06^o}$$

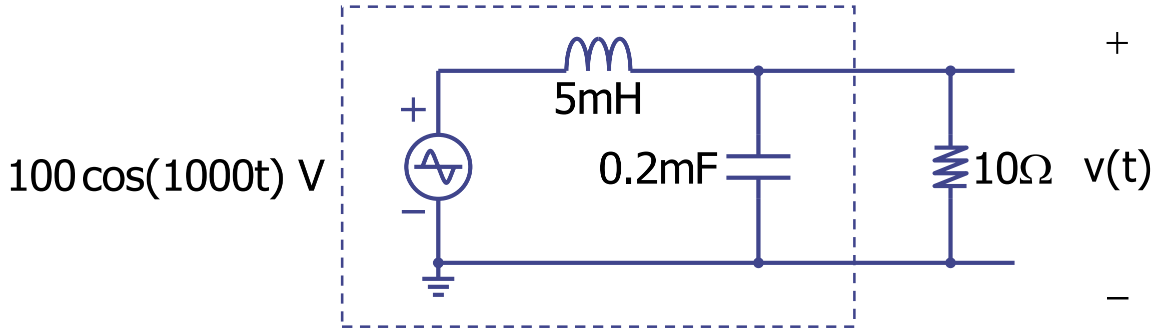
$$= 1.75\angle -29^o$$

$$v_x(t)\big|_{v(t)} = 1.75\cos(3t - 29^o)$$

(3) $$v_x(t) = v_x(t)\big|_{i(t)} + v_x(t)\big|_{v(t)} = 2\cos(2t + 30^o) + 1.75\cos(3t - 29^o)$$

Example 3-28

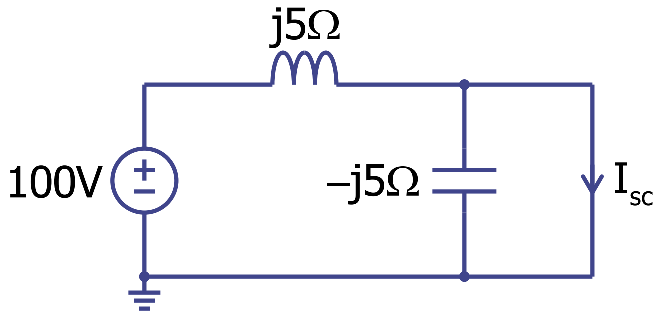
Example 3-28 (Norton's Equivalent): Convert the circuit in the dotted area into its Norton's equivalent circuit and find v(t).



Soln.:

$$Z_\ell = j\omega L = j\times 1000\times 5m = j5\ \Omega$$

$$Z_c = 1/j\omega C = -j/(1000\times 0.2m) = -j5\ \Omega$$

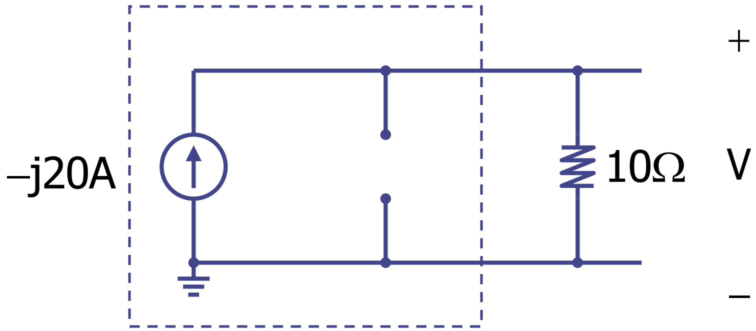


$$I_{sc} = \frac{100}{j5} = -j20A$$

$$Z_{eq} = \frac{j5\times(-j5)}{j5+(-j5)} = \infty\Omega$$

Example 3-28 (cont.)

Use the Norton circuit to compute v(t):



$$V = -j20 \times 10$$

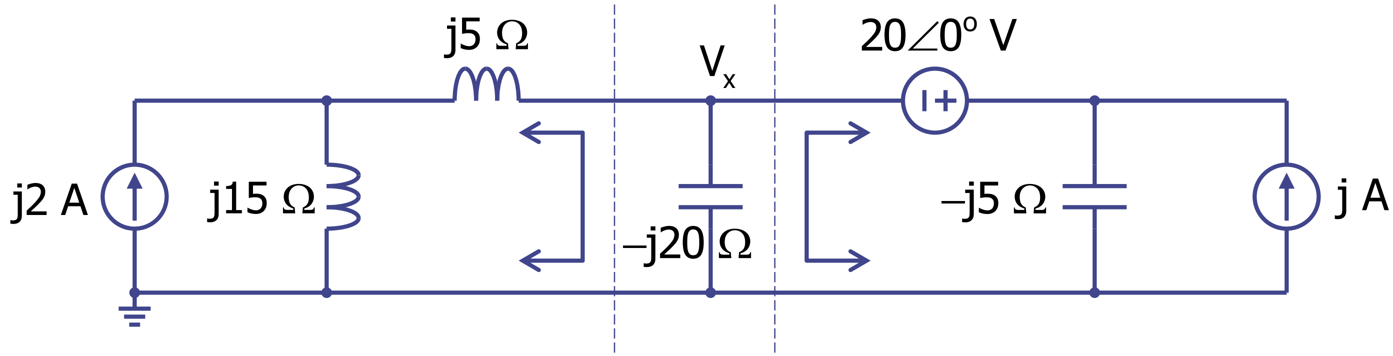
$$= 200\angle -90^o$$

$$\Rightarrow v(t) = 200\cos(1000t - 90^o)\ V$$

Note: The equivalent circuit is only good for one particular frequency!!!

Example 3-29

Example 3-29 (Source Transformation): Find $v_x(t)$.

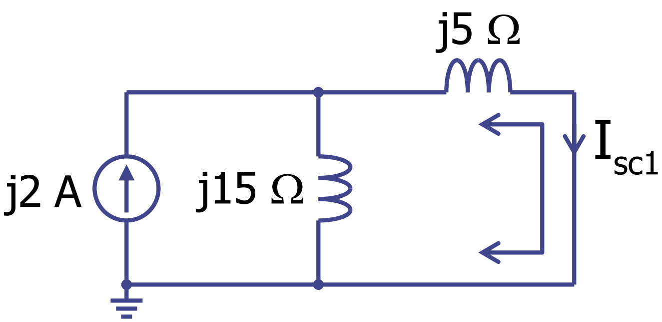


Soln.: There are many ways to solve this problem, and let us work out two Norton's equivalent circuits as shown.

(1) Norton's equivalent of the left circuit:

$$I_{sc1} = \frac{j15}{j15 + j5} j2 = j1.5$$

$$Z_{eq1} = j15 + j5 = j20$$

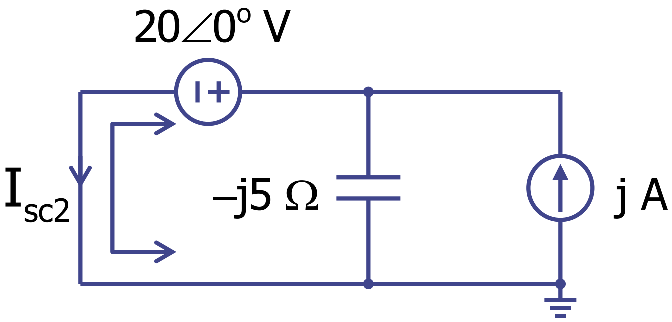


Example 3-29 (cont.)

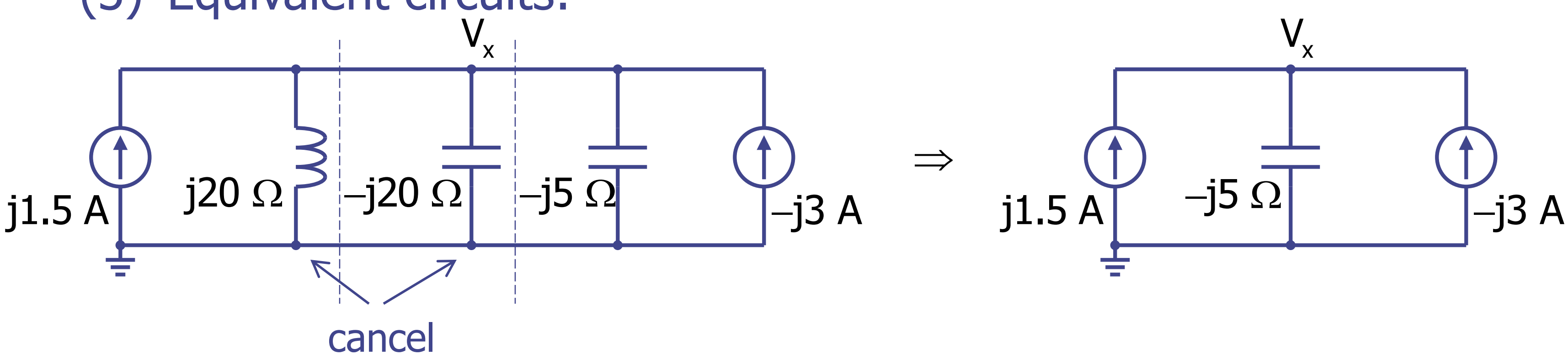
(2) Norton's equivalent of the right circuit:

$$I_{sc2} = j - \frac{20}{-j5} = j - j4 = -j3$$

$$Z_{eq2} = -j5$$



(3) Equivalent circuits:

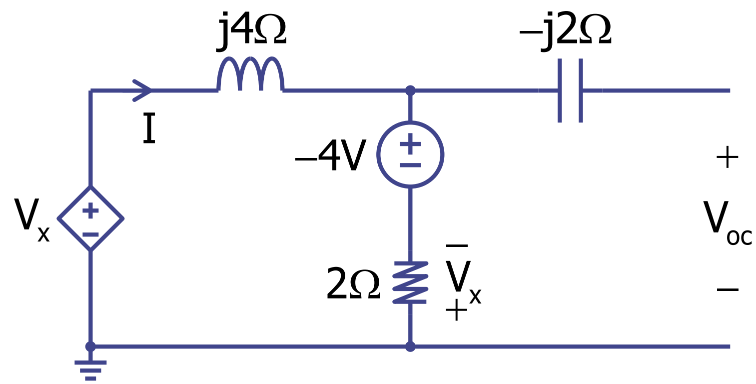


$$V_x = (j1.5 - j3) \times -j5$$

$$= -7.5V$$

Example 3-30

Example 3-30: Find Thevenin's and Norton's equivalent circuits.



Soln.:

Find V_{oc}: For the 2Ω resistor, we have $I \times 2 = -V_x \Rightarrow V_x = -2I$

Next, apply KVL to find I:

$$V_x - I \times j4 - (-4) + V_x = 0$$

$$\Rightarrow \quad 2(-2I) - I \times j4 + 4 = 0 \quad \Rightarrow \quad I = \frac{4}{4 + j4} = \frac{1}{\sqrt{2}\angle 45^o} \text{A}$$

$$\therefore \quad V_{oc} = 2I - 4 = \sqrt{2}\angle(-45^\circ) - 4$$

$$= (1 - j) - 4 = -3 - j$$

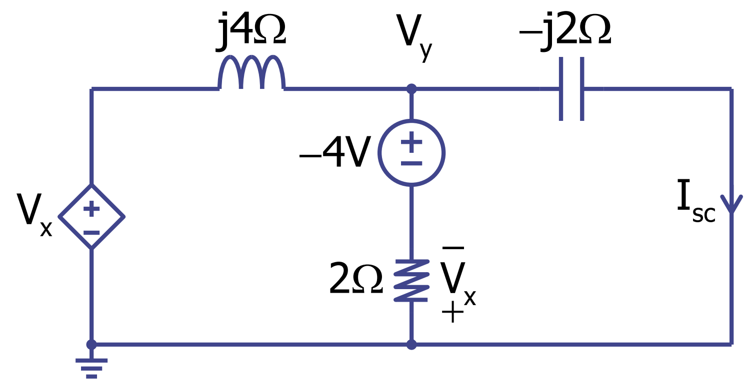
$$= \sqrt{10}\angle(180^\circ + 18.4^\circ) = \sqrt{10}\angle - 161.6^\circ \text{ V}$$

Example 3-30 (cont.)

Find I_{sc}: Method 1 (Nodal Analysis)

First note that

$$V_y = -V_x - 4 \qquad (1)$$



Apply KCL at node V_y :

$$\frac{V_x - V_y}{j4} + \frac{V_x}{2} - \frac{V_y}{-j2} = 0$$

$$V_x - V_y + j2V_x + 2V_y = 0$$

$$V_y = -(1+j2)V_x \qquad (2)$$

From (1) and (2) $\quad V_x + 4 = (1+j2)V_x \quad \Rightarrow \quad V_x = -j2$

From (1) $\quad V_y = -4 + j2$

Hence $\quad I_{sc} = \frac{V_y}{-j2} = -1 - j2 \equiv \sqrt{5}\angle -116.6^\circ$ A

Find Z_{eq}:

$$Z_{eq} = \frac{V_{oc}}{I_{sc}} = \frac{\sqrt{10}\angle -161.6^\circ}{\sqrt{5}\angle -116.6^\circ} = \sqrt{2}\angle -45^\circ = 1 - j\Omega$$

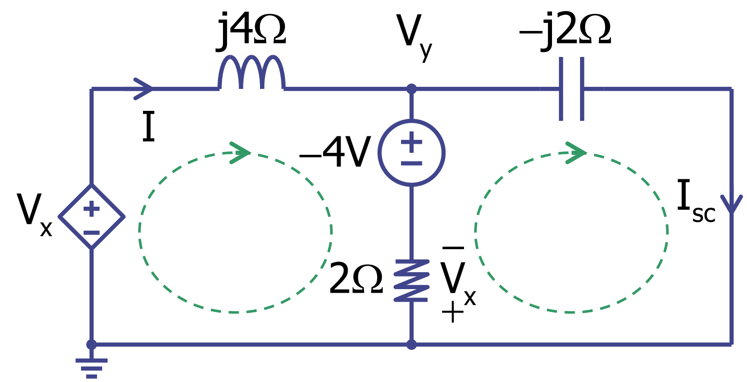
Example 3-30 (Optional)

Find I_{sc}: Method 2 (Mesh Analysis)

KVL on left mesh:

$$V_x - I\times j4 - (-4) + V_x = 0$$

$$\Rightarrow \quad 2V_x - I\times j4 + 4 = 0 \qquad (1)$$



Next, KVL on right mesh:

$$V_x - (-4) + I_{sc}\times(-j2) = 0$$

$$\Rightarrow \quad V_x = I_{sc}\times j2 - 4 \qquad (2)$$

Eliminate V_x from (1) and (2)

$$2V_x = I\times j4 - 4 = 2(I_{sc}\times j2 - 4) = I_{sc}\times j4 - 8$$

$$\Rightarrow \quad I = I_{sc} + j \qquad (3)$$

Since $\quad V_x = (I_{sc} - I)\times 2 = -j2 \qquad$ Voltage across 2 Ω

From (2) $\quad -j2 = I_{sc}\times j2 - 4$

$$\Rightarrow \quad I_{sc} = -1 - j2 = \sqrt{5}\angle -116.6^\circ \text{ A} \qquad (4)$$

Find Z_{eq}:

$$Z_{eq} = \frac{V_{oc}}{I_{sc}} = \frac{\sqrt{10}\angle -161.6^\circ}{\sqrt{5}\angle -116.6^\circ} = \sqrt{2}\angle -45^\circ = 1 - j\Omega$$

3.3.7 AC Power
Root-Mean-Square Value

Root-Mean-Square (RMS) Definition

The Square Root, of the Mean, of the Squared Value, of a Signal.

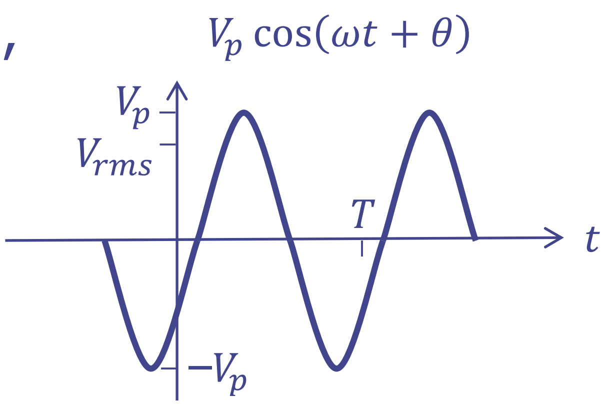
For example, if the signal is $V_p \cos(\omega t + \theta)$, $V_p \geq 0$, then

$$V_{rms} = \sqrt{\frac{1}{T}\int_t^{t+T}\left[V_p \cos(\omega t + \theta)\right]^2 dt}$$

$$= \sqrt{\frac{{V_p}^2}{T}\int_t^{t+T}\frac{1}{2}[1 + \cos(2\omega t + 2\theta)]dt}$$

0

$$= \sqrt{\frac{{V_p}^2}{2T}\int_t^{t+T} dt} = \frac{V_p}{\sqrt{2}} = 0.707 V_p$$



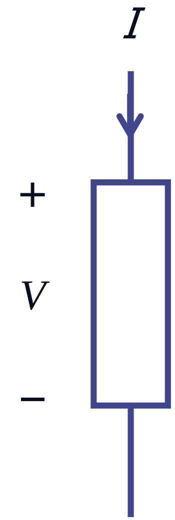
Single-Phase AC Voltage in Hong Kong:

V_{rms}: root-mean-square value = 220 V.

V_p: peak value, amplitude, or magnitude = 311 V.

V_{p-p}: = $2V_p$, peak-to-peak value = 622 V.

Average AC Power



DC Power

$$P = VI$$

Average AC Power

$$P_{ave} = \frac{1}{T}\int_{t}^{t+T} V_p\cos(\omega t+\theta)\times I_p\cos(\omega t+\phi)\,dt$$

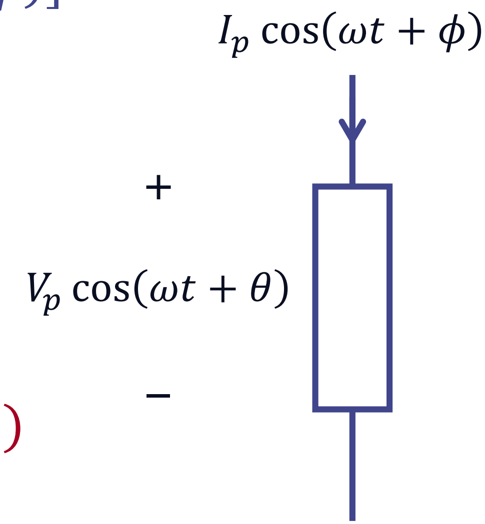
$$= \frac{V_pI_p}{T}\int_{t}^{t+T}\frac{1}{2}[\cos(\theta-\phi)+\cos(2\omega t+\theta+\phi)]dt$$

0

$$= \frac{V_pI_p}{2T}\cos(\theta-\phi)\int_{t}^{t+T}dt$$

$$= \frac{V_pI_p}{2}\cos(\theta-\phi) = \frac{V_pI_p}{2}\cos(\phi-\theta)$$

$$= V_{rms}I_{rms}\cos(\theta-\phi) = V_{rms}I_{rms}\cos(\phi-\theta)$$



$\cos(\theta-\phi)$=$\cos(\phi-\theta)$= Power Factor

Resistor, Capacitor, Inductor Average AC Power

Resistor

$V_p \cos(\omega t + \theta)$ and $I_p \cos(\omega t + \phi)$ are in phase, i.e., $\theta = \phi$.

$$P_{ave} = V_{rms}\, I_{rms} \cos(\phi - \theta) = V_{rms} I_{rms} \cos(0°)$$

$$= V_{rms} I_{rms} = {I_{rms}}^2 R = \frac{{V_{rms}}^2}{R} = \frac{V_p I_p}{2} = \frac{{I_p}^2 R}{2} = \frac{{V_p}^2}{2R}$$

Capacitor

Current $I_p \cos(\omega t + \phi)$ is leading voltage $V_p \cos(\omega t + \theta)$ by 90°, i.e., $\phi = \theta + 90°$.

$$P_{ave} = V_{rms} I_{rms} \cos(\phi - \theta) = \cos(90°) = 0$$

Inductor

Voltage $V_p \cos(\omega t + \theta)$ is leading current $I_p \cos(\omega t + \phi)$ by 90°, i.e., $\theta = \phi + 90°$.

$$P_{ave} = V_{rms} I_{rms} \cos(\phi - \theta) = \cos(-90°) = 0$$

A capacitor or an inductor therefore consumes no *average* power. It stores electrical energy over one half of the period and releases it over the other half.

Complex Power

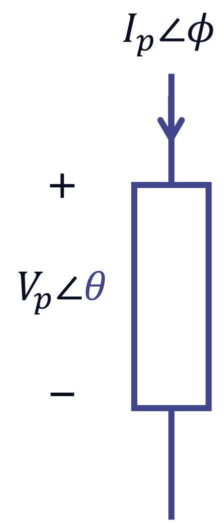
Define complex power in terms of the voltage and current phasors (Remember our phasors represent peak values, not rms).

$$S \equiv \frac{1}{2}\left(V_p\angle\theta\right)\left(I_p\angle\phi\right)^* \quad \text{(* complex conjugate)}$$

$$= \frac{1}{2}\left(V_p\angle\theta\right)\left[I_p\angle(-\phi)\right] = \frac{V_p I_p}{2}\angle(\theta-\phi)$$

$$= \frac{V_p I_p}{2}\cos(\theta-\phi) + j\frac{V_p I_p}{2}\sin(\theta-\phi)$$

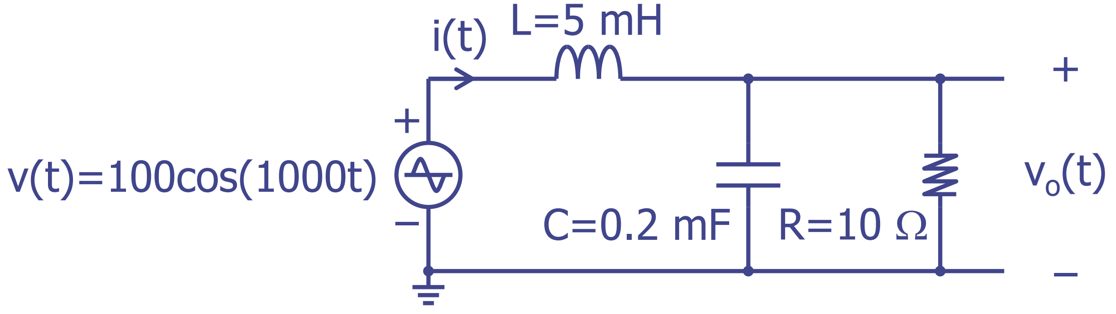
$$= V_{rms}I_{rms}\cos(\theta-\phi) + jV_{rms}I_{rms}\sin(\theta-\phi)$$



Complex Power	Real or Average Power	Reactive or Quadrature Power
(VA, kVA)	(watt, W, kW)	(VA, kVA)

Example 3-31
(From Example 3-22)

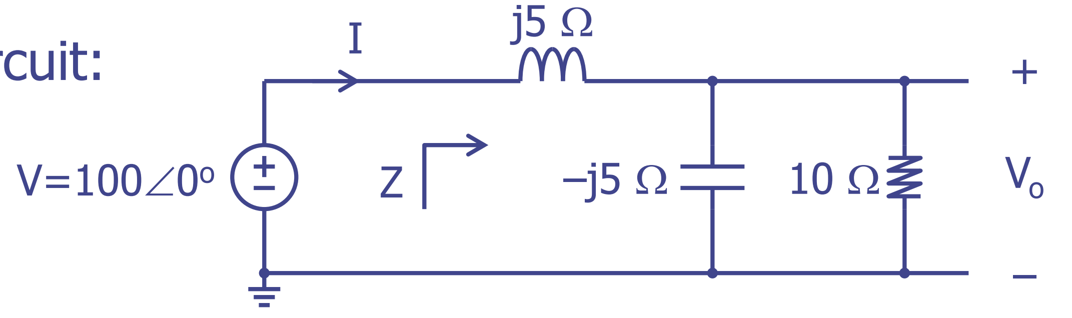
Example 3-31: Compute the average AC power for each circuit element in the following circuit.



Soln.:

We obtained the following results from Example 3-22:

Phasor circuit:



Example 3-31 (cont.)

Voltages and current values from Example 3-22:

$V = 100\angle 0°\ \text{V}$ $\quad V_o = 200\angle(-90°)\ \text{V}$ $\quad I = 44.72\angle(-26.57°)\ \text{A}$

The peak values are:

$V_p = 100\ \text{V}$ $\quad V_{op} = 200\ \text{V}$ $\quad I_p = 44.72\ \text{A}$

Average AC Power:

(1) The inductor and capacitor consume zero average AC power.

(2) For the resistor

$$P_{ave} = \frac{{V_{op}}^2}{2 \times 10\ \Omega} = \frac{(200\ V)^2}{2 \times 10\ \Omega} = 2000\ \text{W}$$

(3) For the voltage source

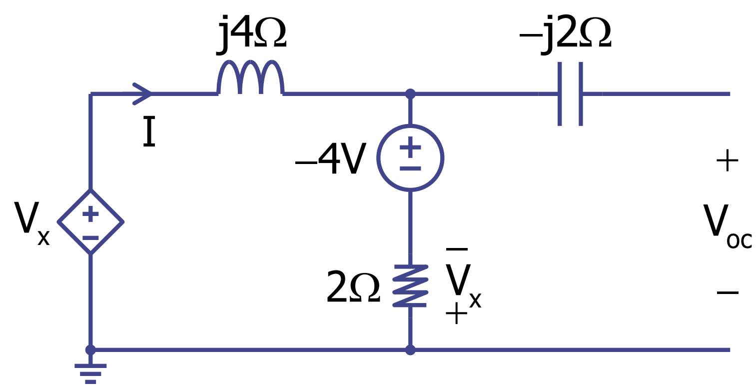
$$P_{ave} = -\frac{V_p I_p}{2}\cos[0° - (-26.57°)]$$

$$= -\frac{100\ V \times 44.72\ A}{2}\cos(26.57°) = -2000\ \text{W}$$

The voltage source is delivering AC power to the resistor.
Conservation of energy holds ☺

Example 3-32
(From Example 3-30)

Example 3-32: Compute the average AC power for each circuit element in the following circuit.



Soln.:

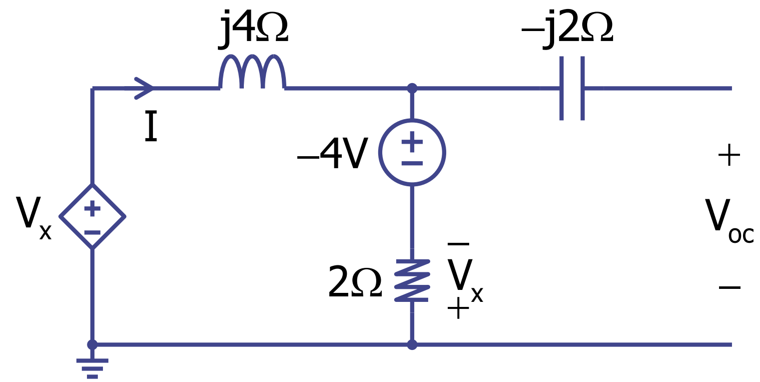
Voltages and current values from Example 3-30:

$$I = \frac{1}{\sqrt{2}}\angle(-45°)\ \text{A} \qquad V_x = -2I = 2\angle(180°)I = \sqrt{2}\angle 135°\ \text{V}$$

The peak values are:

$$I_p = \frac{1}{\sqrt{2}}\ \text{A} \qquad V_{xp} = \sqrt{2}\ \text{V}$$

Example 3-22 (cont.)



Average AC Power:

(1) The inductor and capacitor consume zero average AC power.

(2) For the resistor

$$P_{ave} = \frac{{V_{xp}}^2}{2 \times 2\ \Omega} = \frac{\left(\sqrt{2}\,V\right)^2}{2 \times 2\ \Omega} = 0.5\ \mathrm{W}$$

(3) For the $-4 = 4\angle(180°)$-V independent voltage source

$$P_{ave} = \frac{4I_p}{2}\cos[180° - (-45°)] = \frac{4}{2\sqrt{2}}\cos(225°) = -1\ \mathrm{W}$$

(4) For the dependent voltage source

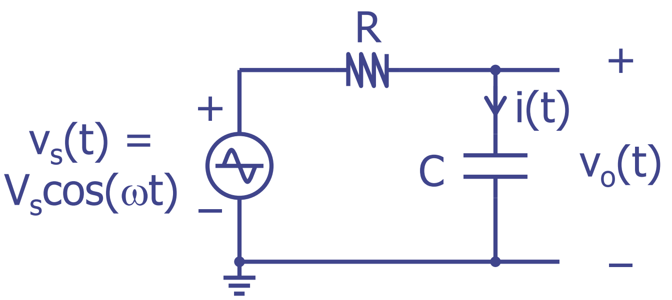
$$P_{ave} = -\frac{V_{xp}I_p}{2}\cos[135° - (-45°)] = -\frac{\sqrt{2}}{2\sqrt{2}}\cos(180°) = 0.5\ \mathrm{W}$$

The independent voltage source is delivering AC power to both the resistor and the dependent voltage source. Again conservation of energy holds ☺

Chapter 3: AC Steady-State Analysis

Appendix: Driving RC Circuit with $V_s\cos(\omega t)$

Consider driving an RC circuit with a sinusoidal voltage source, and KVL gives



$$v_s(t) = Ri(t) + v_o(t)$$

$$\Rightarrow \quad V_s\cos(\omega t) = RC\frac{dv_o(t)}{dt} + v_o(t)$$

$$\Rightarrow \quad \frac{dv_o(t)}{dt} + \frac{1}{\tau}v_o(t) = \frac{V_s}{\tau}\cos(\omega t), \qquad \tau = RC$$

This is a first-order ordinary differential equation. The general solution consists of two parts:

(1) A *general* solution to the homogeneous equation

$$\frac{dv_o(t)}{dt} + \frac{1}{\tau}v_o(t) = 0$$

Rearranging and integrating

$$d\ln\big(v_o(t)\big) = \frac{dv_o(t)}{v_o(t)} = -\frac{dt}{\tau}, \qquad \ln\big(v_o(t)\big) = -\frac{t}{\tau} + K'$$

we obtain the Part 1 solution: $\boxed{v_o(t) = Ke^{-t/\tau}}$

Appendix: Driving RC Circuit with $V_s\cos(\omega t)$ (2)

(2) A *particular* solution to the original differential equation

$$\frac{dv_o(t)}{dt} + \frac{1}{\tau} v_o(t) = \frac{V_s}{\tau}\cos(\omega t) \qquad (1)$$

for which, an educated guess is

$$v_o(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\Rightarrow \qquad \frac{dv_o(t)}{dt} = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)$$

Substituting back into (1) gives

$$-A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t) + \frac{A}{\tau}\cos(\omega t) + \frac{B}{\tau}\sin(\omega t) = \frac{V_s}{\tau}\cos(\omega t)$$

This must be true for all t. As $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent of each other, the only way this can happen is when the coefficients of the $\sin(\omega t)$ terms are equal on both sides of the equation. Same must also be true for the $\cos(\omega t)$ terms.

Appendix: Driving RC Circuit with $V_s\cos(\omega t)$ (3)

Matching the $\sin(\omega t)$ terms:

$$-A\omega + \frac{B}{\tau} = 0 \qquad \Rightarrow \qquad B = A\omega\tau$$

Matching the $\cos(\omega t)$ terms:

$$B\omega + \frac{A}{\tau} = \frac{V_s}{\tau} \qquad \Rightarrow \qquad A\omega^2\tau + \frac{A}{\tau} = \frac{V_s}{\tau}$$

Giving

$$A = \frac{1}{1+\omega^2\tau^2}V_s \quad \text{and} \quad \mathrm{B} = \frac{\omega\tau}{1+\omega^2\tau^2}V_s$$

The Part 2 solution is thus given by

$$v_o(t) = \frac{1}{1+\omega^2\tau^2}V_s\cos(\omega t) + \frac{\omega\tau}{1+\omega^2\tau^2}V_s\sin(\omega t)$$

Appendix: Driving RC Circuit with $V_s\cos(\omega t)$ (4)

However, we would like to turn $v_O(t)$ into a cosine function only so that it can be compared to the input voltage $V_s\cos(\omega t)$. The standard intermediate procedure goes as follows:

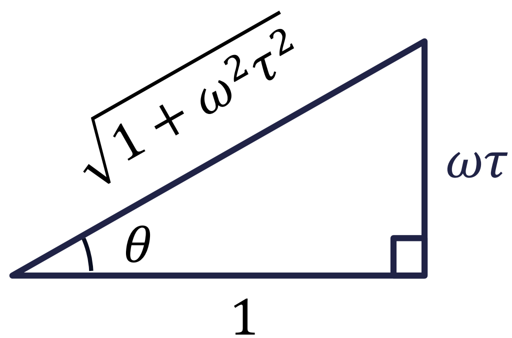
$$v_o(t) = \frac{1}{1+\omega^2\tau^2}V_s\cos(\omega t) + \frac{\omega\tau}{1+\omega^2\tau^2}V_s\sin(\omega t)$$

$$= \frac{V_s}{\sqrt{1+\omega^2\tau^2}}\frac{1}{\sqrt{1+\omega^2\tau^2}}\cos(\omega t) + \frac{V_s}{\sqrt{1+\omega^2\tau^2}}\frac{\omega\tau}{\sqrt{1+\omega^2\tau^2}}\sin(\omega t)$$

$$= \frac{V_s}{\sqrt{1+\omega^2\tau^2}}\cos\theta\cos(\omega t) + \frac{V_s}{\sqrt{1+\omega^2\tau^2}}\sin\theta\sin(\omega t)$$

$$= \frac{V_s}{\sqrt{1+\omega^2\tau^2}}\cos(\omega t - \theta)$$

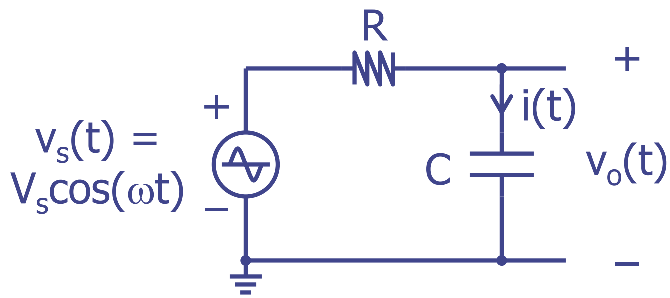
where $\theta = \tan^{-1}(\omega\tau)$



This is the Part 2 solution in its final form!

Appendix: Driving RC Circuit with $V_s\cos(\omega t)$ (5)

Going back to the original problem of driving an RC circuit with a sinusoidal voltage source, the general expression for the output voltage is finally obtained by combining the two-part solutions obtained earlier:



$$v_o(t) = Ke^{-t/\tau} + \frac{V_s}{\sqrt{1+\omega^2\tau^2}}\cos(\omega t - \theta)$$

transient response | steady-state AC response

The first term is a transient response that decays exponentially with time according to $\tau = RC =$ time constant. The transient only lasts for a few time constants. We will deal with transients in our later chapter.

The second term is the steady-state AC response. This is what we are currently interested in.

The constant K can be determined by the initial condition, i.e., $v_o(0)$.